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SOME ELEMENTARY CRYPTANALYSIS OF ALGEBRAIC CRYPTOGRAPHY

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1. Introduction. By algebraic cryptography we mean the process of encipherment which converts a plain message into a cipher message by means of n simultaneous linear congruences, where n is an arbitrary integer ([1], [2], [3], [4]). The plain letters are written in blocks of n, and if $P_{i\beta}$ denotes the letter in position β of block i, then

$$C_{i\beta} \equiv a_{\beta 1}P_{i1} + \cdots + a_{\beta n}P_{in} \pmod{26}, \tag{1.1}$$

where $i = 1, 2, \cdots, \beta = 1, \cdots, n$, so that the encipherment of plain block $P_{i1} \cdots P_{in}$ is cipher block $C_{i1} \cdots C_{in}$.

In (1.1) the matrix $A = [a_{\alpha\beta}]$ of the coefficients is such that $|a_{\alpha\beta}|$ is prime to 26. It is convenient for cryptographic purposes to take A such that $A = A^{-1}$, and we assume A is so chosen. Hence we can also write

$$P_{i\beta} \equiv a_{\beta 1}C_{i1} + \cdots + a_{\beta n}C_{in} \pmod{26}. \tag{1.2}$$

The 26 letters of the alphabet are given numerical values according to some permutation of the normal sequence 0, 1, $\cdots$, 25; and it is these numerical values which are actually used for the $P_{i\beta}$ and $C_{i\beta}$ above. For the purposes of this article there is no loss of generality in using the normal sequence itself, giving

(1.3)

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

For illustrations in the actual use of (1.1), (1.3), [3] may be consulted.

In its most general form the cryptanalytic problem involved here may be stated in the following way. Given one (or more) cryptograms obtained by use of (1.1), but with no other information, to find the corresponding plain messages, and the matrix A. Even for such small values of n as $n = 5$ this would ordinarily be a problem of very great difficulty.

Our purpose here is to treat what may be considered as one of the simplest special cases of the general problem. We assume as known the numerical values of the alphabet letters (see (1.3)), and also as known some portion of the plain message of a cryptogram. However, the exact location of this portion in the message is not known. This is an instance of the classical "method of the probable word," well known in the cryptanalytic art. The solution of this case involves two steps:

Problem (A): Determining the location of the known plain-text.

Problem (B): Determining matrix A.

The final test is of course deciphering the cryptogram based on the recovered matrix.

2. Some basic relations. Consider any $n+1$ blocks of plain-text $P_{i1} \cdots P_{in}$ for $i=i_1, \cdots, i_{n+1}$. By (1.2) we can write

$$P_{i_t\beta} \equiv \sum_{\sigma=1}^{n} a_{\beta\sigma} C_{i_t\sigma} \qquad (t=1, \cdots, n+1), \tag{2.1}$$

where in (2.1), and hereafter unless otherwise stated, all congruences are taken mod 26. In (2.1) consider β as fixed (a value from $1, \cdots, n$). Regarding (2.1) as a system of $n+1$ congruences in n unknowns $a_{\beta 1}, \cdots, a_{\beta n}$, we may eliminate these to obtain the condition

$$\Delta_\beta = \begin{vmatrix} P_{i_1\beta} & C_{i_1 1} & \cdots & C_{i_1 n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ P_{i_{n+1}\beta} & C_{i_{n+1}\beta} & & C_{i_{n+1}n} \end{vmatrix} \equiv 0 \qquad (\beta = 1, \cdots, n). \tag{2.2}$$

In a like manner we derive from (1.1),

$$\Delta'_\beta = \begin{vmatrix} C_{i_1\beta} & P_{i_1 1} & \cdots & P_{i_1 n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ C_{i_{n+1}\beta} & P_{i_{n+1}1} & \cdots & P_{i_{n+1}n} \end{vmatrix} \equiv 0 \qquad (\beta = 1, \cdots, n). \tag{2.3}$$

Relations $\Delta_\beta \equiv 0$, $\Delta'_\beta \equiv 0$ will be found useful in our problem since they do not involve the (unknown) matrix elements $a_{\alpha\beta}$. Their use is illustrated below.

From (2.1) select any n values of i_t, say, $j_1, \cdots, j_n$, and form the determinant based on the resulting n^2 congruences (using $\beta=1, \cdots, n$). This gives

$$|P_{j_t\beta}| \equiv |\sum a_{\beta\sigma} C_{j_t\sigma}| \equiv |a_{\beta\sigma}| \, |C_{j_t\sigma}|,$$

or

$$\begin{vmatrix} P_{j_1 1} & \cdots & P_{j_1 n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ P_{j_n 1} & \cdots & P_{j_n n} \end{vmatrix} \equiv \pm \begin{vmatrix} C_{j_1 1} & \cdots & P_{j_1 n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ C_{j_n 1} & \cdots & C_{j_n n} \end{vmatrix} \tag{2.4}$$

since $|a_{\alpha\beta}| \equiv \pm 1$, as follows from $A = A^{-1}$.

In (2.2), (2.3), (2.4), the cipher values are known. The plain values will be selected from the known probable text. Briefly, the correct location of this known text will be found by fitting it in all positions until one is obtained for which (2.2), (2.3), (2.4) are satisfied. The exact method for doing this with a minimum of trials is explained in the sections to follow.

3. Problem (A) and the use of modulo 2. Consider the system of n congruences (1.1) for a fixed i and for modulo 2. We write these as

$$C_\beta \equiv a_{\beta 1} P_1 + \cdots + a_{\beta n} P_n \pmod 2 \qquad (\beta = 1, \cdots n) \tag{3.1}$$

so that $P_1 \cdots P_n$, $C_1 \cdots C_n$ represent any block of n plain and corresponding

n cipher values. As $P_1 \cdots P_n$ assumes all possible sequences of n letters, (3.1) establishes a unique 1-1 reciprocal correspondence between the 2^n length n binary sequences of 0's and 1's. These 2^n sequences represent the 2^n integers (in base 2) from 0 through 2^n-1. Hence by (3.1) there is associated with each block of n plain letters and its corresponding block of n cipher letters a pair of numbers in the range 0 through 2^n-1. The number thus associated with a block of n letters is called its binary value.

The following example illustrates these pairings and their use in Problem (A).

(a_1)	CRY	PTO	GRA	PHY	BYA	LGE	BRA	ICE	QUA	TIO	NSX
(b_1)	SUI	RIM	AYG	DIK	VFG	LTE	RUK	KRC	QHA	JLY	JOB

(a_2)	3	18	25	16	20	15	7	18	1	16	8	25	2	25	1	12	7	5	2	18	1
(b_2)	19	21	9	18	9	13	1	25	7	4	9	11	22	6	7	12	20	5	18	21	11

(a_2)	9	3	5	17	21	1	20	9	15	14	19	24
(b_2)	11	18	3	17	8	1	10	12	25	10	15	2

(a_3)	101	001	101	001	011	011	001	111	111	011	010
(b_3)	111	011	111	011	001	001	011	101	101	001	010

(a_4)	5	1	5	1	3	3	1	7	7	3	2
(b_4)	7	3	7	3	1	1	3	5	5	1	2

Here, (a_1), (b_1) represent a plain-text with its cipher text; (a_2), (b_2) are the numerical values of the letters by (1.3); (a_3), (b_3) give these values (mod 2); and (a_4), (b_4) are the binary values according to the scheme

$$000 = 0,\ 000 = 1,\ 010 = 2,\ 011 = 3,\ 100 = 4,\ 101 = 5,\ 110 = 6,\ 111 = 7.$$

Thus, the binary value of CRY = 5, of SUI = 7, etc. Because of $A = A^{-1}$ the binary value pairing as in (a_4), (b_4), will always be such that $a \leftrightarrow b$, $0 \leftrightarrow 0$ where a may equal b. (In the above example it is seen that $1 \leftrightarrow 3$, $2 \leftrightarrow 2$, $5 \leftrightarrow 7$). It follows that the two binary-value sequences derived from any plain-text and its corresponding cipher text must always be of the same pattern. This pattern in the present illustration could be represented by the sequence *ababccbddce* (see (a_4), (b_4)). Furthermore, any portion of the plain-text must produce a common pattern with its associated cipher portion.

It is this last property which is used to obtain preliminary locations (apparent settings) of the probable text. The cipher text is converted to its binary values (as in (b_4) above), and the probable text is similarly converted, assuming it starts in each of the n positions of a block. There will thus be n such binary conversions, $B_1, \cdots, B_n$, each of which is then matched against the cipher text conversion for like patterns. Each such matching is an apparent setting of the probable text, and these are next tested against conditions (2.2), (2.3), (2.4), depending on the material available. This feature is taken up in the next section.

To demonstrate the pattern matching, consider the example,

	MIU	GNJ	WWU	YHZ	DNS	WVK	RFV	LLK	AMP	IGS	MIU
	7	4	7	4	1	5	0	1	6	7	7
(3.2)	WKN	OEM	IEK	ORW	WAE	KZB	APL	KYP	MEU	ZMO	ZIX
	6	7	7	5	7	4	4	6	7	3	6
	FHS	SJI	DDJ	KFY	BWW	HQP	KLI	NKG	TMJ	ROB	TZE
	1	5	0	5	3	2	5	3	2	2	1

The number under each group is its binary value. We assume the probable text THREE CONGRUENCES, but use the first 14 letters only. Let S_α indicate the first letter starts in position α of a block ($\alpha = 1, 2, 3$). We then obtain

S_1:	THR	EEC	ONG	RUE	NCx	S_2:	xTH	REE	CON	GRU	ENC
B_1:	0	7	5	3		B_2:		3	6	5	5

S_3:	xxT	HRE	ECO	NGR	UEN	Cxx
B_3:		1	7	2	6	

From (3.2) the cipher text binary sequence is

(3.3) 7 4 7 4 1 5 0 1 6 7 7 6 7 7 5 7 4 4 6 7 3 6 1 5 0 5 3 2 5 3 2 21

The B_1 pattern, 0 7 5 3, must obviously match at 0 1 6 7 or 0 5 3 2 of (3.3), giving

(3.3)	0 1 6 7	0 5 3 2
B_1:	0 7 5 3	0 7 5 3

Each of these matchings can be eliminated by inspection, the first by 7 paired with both 1 and 3, and the second by 3 with 2 and 5.

The B_2 pattern, 3 6 5 5, results in four apparent settings:

(3.3)	1 6 7 7	7 6 7 7	5 7 4 4	5 3 2 2
B_2:	3 6 5 5	3 6 5 5	3 6 5 5	3 6 5 5

of which all but the first can be eliminated at once. The first setting is kept for further testing.

The B_3 pattern is matched at

(3.3)	7 4 1 5	4 6 7 3	7 3 6 1	3 6 1 5
B_3:	1 7 2 6	1 7 2 6	1 7 2 6	1 7 2 6

all of which are inconsistent. If the probable text is actually present, then it must start in block 7 of (3.2):

		1	6	7	7
(3.4)	RFV	LLK	AMP	IGS	MIU
	xTH	REE	CON	GRU	ENC
		3	6	5	5

4. Problem (A) continued. Testing of apparent settings. In the above example only one apparent setting, (3.4), survived the binary pattern test. Ordinarily with longer messages, shorter probable texts, or larger values of n, a large number of apparent settings will remain, and these must be further eliminated.

Let λ equal the length of the probable text. Then the actual procedure used will largely depend on the value of λ.

Suppose first $\lambda \geqq n(n+1)+(n-1)$. For any starting point S_α write the probable text corresponding to an apparent setting beginning in block i and position α in an array of consecutive blocks vertically as given below (corresponding cipher blocks are also shown).

$$
\begin{array}{l|l}
x \cdots \; x\, P_{i\alpha} \quad \cdots P_{in} & C_{i1} \quad \cdots C_{in} \\
P_{i+1,1} \quad \cdots P_{i+1,\alpha} \quad \cdots P_{i+1,n} & C_{i+1,1} \quad \cdots C_{i+1,n} \\
\vdots \qquad \vdots \qquad \vdots & \vdots \qquad \vdots \\
P_{i+n+1,1} \cdots P_{i+n+1,\alpha} \cdots P_{i+n+1,n} & C_{i+n+1,1} \cdots C_{i+n+1,n}
\end{array} \tag{4.1}
$$

Then regardless of the values of i and α there will always be at least $n+1$ complete blocks of (presumably) known plain-text as indicated in (4.1). Now if there are a large number of apparent settings with a common value of α, it will be found that (2.3), or $\Delta_\beta' \equiv 0$, is best to use to test these settings. For n columns of (2.3) can be kept fixed, these being selected from the array of plain-text columns in (4.1). The remaining column is chosen from the cipher-text array at the right in (4.1), and (2.3) then expanded using cofactors of this C column,

$$P_1 C_{i\beta} + \cdots + P_{n+1} C_{i+n+1,\beta} \equiv 0. \tag{4.2}$$

The cofactors $P_1, \cdots, P_{n+1}$ being independent of i need be calculated only once as i varies through values corresponding to the apparent settings.

If (4.2) be first tested (mod 2), many apparent settings will be eliminated very easily. The remaining are then tested (mod 26).

If we apply these ideas to the apparent setting of (3.4) we have corresponding to (4.1),

$$
\begin{array}{l|l}
x\ \mathrm{T}\ \mathrm{H} & \mathrm{R}\ \mathrm{F}\ \mathrm{V} \\
\mathrm{R}\ \mathrm{E}\ \mathrm{E} & \mathrm{L}\ \mathrm{L}\ \mathrm{K} \\
\mathrm{C}\ \mathrm{O}\ \mathrm{N} & \mathrm{A}\ \mathrm{M}\ \mathrm{P} \\
\mathrm{G}\ \mathrm{R}\ \mathrm{U} & \mathrm{I}\ \mathrm{G}\ \mathrm{S} \\
\mathrm{E}\ \mathrm{N}\ \mathrm{C} & \mathrm{M}\ \mathrm{I}\ \mathrm{U}
\end{array} \tag{4.3}
$$

Forming Δ_1' gives

$$
\begin{vmatrix}
18 & 5 & 5 & 12 \\
3 & 15 & 14 & 1 \\
7 & 18 & 21 & 9 \\
5 & 14 & 3 & 13
\end{vmatrix},
$$

which is $\equiv 0 \pmod 2$ and $\pmod{26}$. It is found that Δ_2', Δ_3' are also both $\equiv 0 \pmod 2$ and $\pmod{26}$. In addition, (2.4) is satisfied for any choice of 3 rows from (4.3). This setting would next be used to obtain matrix A, (Problem (B)).

Suppose next $n^2 < \lambda < n(n+1)+n-1$. In this case certain starting points will not produce $n+1$ complete plain-text blocks in array (4.1), and the method described above will consequently fail. For such cases we can use (2.2), (and (2.4) when available), since, if $\lambda > n^2$, there must always be at least one column in the plain-text array of (4.1) containing $n+1$ letters, this column then being used in (2.2). Thus, for certain settings (values of α), (2.2) is used, and for the remaining (2.3) is used. In addition (2.4) is used when necessary.

We discuss again the previous example (3.2), this time using only 10 letters of the probable text (this being the minimum for $n=3$), or THREE CONGR. The three starting points give

S_1	S_2	S_3
T H R 0	x T H 0/4	x x T
E E C 7	R E E 3	H R E 1
O N G 5	C O N 6	E C O 7
R x x	G R x 4/5	N G R 2

The binary patterns are written at the right of each array. In S_2 we indicate the two possibilities 0/4, 4/5 as shown. These patterns are now matched against the cipher pattern (3.3), with the following preliminary matchings:

B_1: 0 7 5 0 7 5

0 1 6 0 5 2

B_2: 0/4 3 6 4/5	B_3: 1 7 2
7 4 1 5	7 4 1
0 1 6 7	4 1 5
4 6 7 3	1 6 7
6 7 3 6	5 7 4
7 3 6 1	4 6 7
3 6 1 5	6 7 3
0 5 3 2	7 3 6
5 3 2 5	3 6 1
3 2 5 3	6 1 5
2 5 3 2	5 3 2
	2 5 3
	5 3 2

All but the following seven are eliminated by inspection,

0 7 5	0 3 6 4/5	4 3 6 5	4 3 6 4	1 7 2	1 7 2	1 7 2
0 1 6	0 1 6 7	7 3 6 1	5 3 2 5	5 7 4	6 7 3	5 3 2
$S_1(a)$	$S_1(b)$	$S_2(c)$	$S_2(d)$	$S_3(e)$	$S_3(f)$	$S_3(g)$

The corresponding arrays (4.1) are

T H R	R F V	x T H	R F V	M E U	K F Y	x x T	I E K	A P L	D D J
E E C	L L K	R E E	L L K	Z M O	B W W	H R E	O R W	K Y P	K F Y
O N G	A M P	C O N	A M P	Z I X	H Q P	E C O	W A E	M E U	B W W
R x x	I G S	G R x	I G S	F H S	K L I	N G R	K Z B	Z M O	H Q P
	$S_1(a)$		$S_2(b)$	$S_2(c)$	$S_2(d)$		$S_3(e)$	$S_3(f)$	$S_3(g)$

Since none of the S_α (plain-text) settings contain four complete blocks we must use text (2.2), first (mod 2). The column TEOR of length 4, from the probable text, is of course used as the plain-text column in (2.2) in all cases. Cases (c), (e), (f) are eliminated using modulo 2 and (2.2). The remaining four are tested (mod 26). This eliminates (d) and (g), leaving $S_1(a)$ and $S_2(b)$. We now use (2.4) on $S_1(a)$ as three complete blocks are present in the probable text, and this eliminates $S_1(a)$.

$S_2(b)$ is the only case left ((2.4) cannot be used on it) and the next step would be to recover matrix A. This is taken up in the next section.

5. Problem (B). Determination of matrix A. Assuming a probable text has been located (as in $S_2(b)$ above), (1.2) give a series of n^2 congruences for the determination of the elements $a_{\alpha\beta}$ of matrix A. For a fixed α, (1.2) determine at least n congruences in the n unknowns $a_{\alpha 1}, \cdots, a_{\alpha n}$. If their coefficient matrix contains an $n \times n$ determinant prime to 26, all these unknowns are determined uniquely. Otherwise, there will be several solutions possible. Finally the use of condition $A^2 = I$ will pick out the correct matrix which should decipher the cryptogram.

We carry out this procedure by completing the solution of above example (3.2). The setting $S_2(b)$ is to be used with (1.2). Convert all letters to numerical values, and write the corresponding (1.2) in matrix form:

$$(5.1) \qquad \begin{bmatrix} x & 18 & 3 & 7 \\ 20 & 5 & 15 & 18 \\ 8 & 5 & 14 & x \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 18 & 12 & 1 & 9 \\ 6 & 12 & 13 & 7 \\ 22 & 11 & 16 & 19 \end{bmatrix}$$

or, $P \equiv AC$, the various rows of $S_2(b)$ now appearing as columns. The C matrix contains no 3×3 determinant prime to 26. The 10 congruences of (5.1) split into 3 sets of 3, 4, 3 respectively for the 9 unknowns $a_{\alpha 1}, a_{\alpha 2}, a_{\alpha 3}$ ($\alpha = 1, 2, 3$).

Solving the congruences of (5.1) using mod 2 and mod 13 gives

$$(5.2) \qquad \begin{aligned} a_{11} &= 7, \quad a_{12} = 8, \quad a_{13} = 16, \text{ or} \\ a_{11} &= 20, \quad a_{12} = 21, \quad a_{13} = 16; \end{aligned}$$

$$(5.3)\qquad \begin{aligned} a_{31} &= 6, \quad a_{32} = 9, \quad a_{33} = 3, \text{ or} \\ a_{31} &= 19, \quad a_{32} = 22, \quad a_{33} = 3. \end{aligned}$$

The 3 congruences for a_{31}, a_{32}, a_{33} have a zero determinant, so it will be simpler to use condition $A^2 = I$, taking each of the four pairs of solutions from (5.2), (5.3) for the first two rows of A. The choice

$$\begin{bmatrix} 7 & 8 & 16 \\ 6 & 9 & 3 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 7 & 8 & 16 \\ 6 & 9 & 3 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is consistent, with $a_{31} = 20$, $a_{32} = 18$, $a_{33} = 11$. The other three cases give contradictions. We finally have then for matrix A,

$$A = \begin{bmatrix} 7 & 8 & 16 \\ 6 & 9 & 3 \\ 20 & 18 & 11 \end{bmatrix},$$

giving $P_1 = 7C_1 + 8C_2 + 16C_3$, $P_2 = 6C_1 + 9C_2 + 3C_3$, $P_3 = 20C_1 + 18C_2 + 11C_3$ to be used in deciphering the cryptogram. This is left for the reader.

6. Concluding remarks. In case the length of the probable text is $\lambda \leqq n^2$, the above methods are in general unavailable. Even assuming a correct location would involve a considerable amount of work, too lengthy to be discussed here. If two or more short probable texts are known, the simultaneous testing of assumed locations may enable the present methods to be used.

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THE BEHAVIOR OF ENTIRE FUNCTIONS AND A CONJECTURE OF ERDÖS

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1. Introduction. Let $f(z)=\sum_0^\infty a_n z^n$ be an entire function. The maximum term of $f(z)$ is defined to be $\max_n |a_n| r^n$ and is denoted by $\mu(r, f)$, or for brevity, by $\mu(r)$. The largest value of n such that $\mu(r)=|a_n| r^n$ is denoted by $\nu(r, f)$ or $\nu(r)$. It is known ([4], p. 32) that

$$\mu(r) < M(r) < \mu(r)\left\{2\nu\left(r+\frac{r}{\nu(r)}\right)+1\right\}, \tag{1.1}$$

where $M(r)=M(r, f)=\max_{|z|=r}|f(z)|$. The following theorem was proved by J. Clunie [1].

THEOREM A. *Let $f(z)=\sum_1^\infty a_n z^n$ be an entire function and β be a positive number. If*

$$\left(\frac{n+1}{n}\right)^\beta\left|\frac{a_n}{a_{n+1}}\right| > \left(\frac{n}{n-1}\right)^\beta\left|\frac{a_{n-1}}{a_n}\right| \tag{1.2}$$

for all sufficiently large n, then

$$M(r, f) < \{1+o(1)\}\beta^{-\beta-1}\Gamma(1+\beta)e^\beta \nu(r, f)\mu(r, f). \tag{1.3}$$

In the following theorem, to be proved in Section 3, we suppose that the coefficients (a_n) satisfy somewhat less restrictive conditions and arrive at an extension of Clunie's theorem.

THEOREM 1. *Let $f(z)=\sum_1^\infty a_n z^n$ be an entire function satisfying the following conditions*:

(i) $$\max_{0\le\theta\le 2\pi}\left|\sum_1^N \exp(in\theta + i\arg a_n)\right| \le CN^\alpha,$$

where α and C are constants, $\frac{1}{2}\le\alpha\le 1$, $C>0$, for $N=1, 2, \cdots$; (ii) $|a_{n-1}/a_n|$ is strictly increasing for $n>n_0$; (iii) $|a_{n-1}/a_n| \ge |a_{p-1}/a_p|\{n/(n-1)\}^\alpha$, $p=[nC_1]$, where C_1 is a constant such that $0<C_1<1$, for $n>n_0$. Then

$$M(r, f) < C\{1+2C_1^{-\alpha}+o(1)\}\mu(r, f)(\nu(r, f))^\alpha. \tag{1.4}$$

COROLLARY. *If $f(z)=\sum_1^\infty a_n z^n$ is an entire function satisfying conditions* (ii) *and* (iii) *with $\alpha=1$, then*

$$M(r, f) < \{1+2C_1^{-1}+o(1)\}\mu(r, f)\nu(r, f). \tag{1.5}$$

We note that our hypothesis on $|a_n/a_{n+1}|$ is less restrictive than (1.2), since (1.2) implies (ii) and (iii) with $0<C_1<\beta/(\beta+\alpha)$, whereas (1.2) does not follow from

* I am indebted to the referee for helpful comments.

conditions (ii) and (iii). Furthermore, (1.4) gives a sharper result than (1.3) when $\alpha<1$; our proof will not depend upon the Wiman-Valiron theory ([4], [6]). Since (1.3), (1.4) and (1.5) are asymptotic inequalities, they hold also when $f(0)\neq 0$.

2. Conjecture of Erdös. Paul Erdös ([2], problem 25) conjectured that if for an entire function $f(z)$, $\lim_{r\to\infty}\mu(r,f)/M(r,f)$ exists, then this limit must be equal to zero. We prove this conjecture when the coefficients (a_n) satisfy a certain condition (Th. 2 (i)) or when $f(z)$ can be written as a product of two entire functions $F(z)$ and $G(z)$ satisfying certain conditions (Th. 3). We also obtain inequalities for $U(f)=\limsup_{r\to\infty}\mu(r,f)/M(r,f)$ and $u(f)=\liminf_{r\to\infty}\mu(r,f)/M(r,f)$. Let us define

$$L=L(f)=\limsup_{n\to\infty}\left|a_n^2/(a_{n-1}a_{n+1})\right|;\qquad l=l(f)=\liminf_{n\to\infty}\left|a_n^2/(a_{n-1}a_{n+1})\right|.$$

THEOREM 2. *Let $f(z)=\sum_0^\infty a_nz^n$ be an entire function satisfying* (ii) *of Theorem* 1. (i) *If $L=1$, then $U(f)=0$;* (ii) *if $L>1$, then $U(f)\geq(\sqrt{L}-1)/(\sqrt{L}+1)$;* (iii) *if $l>1$, then $u(f)\geq\{2+3l^{-1}+(2/(l^3-1))\}^{-1}$;* (iv) *if $l=\infty$, then $u(f)=\frac{1}{2}$;* (v) *if $L=\infty$, then $U(f)=1$.*

We note that the condition $l>1$ implies condition (ii) of Theorem 1.

THEOREM 3. *Let $f(z)=F(z)G(z)$, where $F(z)$ and $G(z)$ are entire functions satisfying the following conditions:*

(i) $M(r,F)M(r,G)=O(M(r,f))$,

(ii) $\mu(r,F)/M(r,F)=O(r^{-\alpha})$, $\alpha>0$,

(iii) $M(ar,G)/M(r,G)=o(r^{\alpha})$, $a>1$,

as r tends to infinity. Then $U(f)=0$. If any one of the three conditions holds for a sequence of values of $r\uparrow\infty$, and the remaining two conditions for $r\to\infty$, then $u(f)=0$.

THEOREM 4. *Let $f(z)$ be an entire function* of mean type of integer order ρ such that the canonical product formed with zeros of $f(z)$ is at most zero type of order ρ. Let $\phi(z)$ be any entire function such that $M(r,\phi)=o(M(r,f))$. Then $U(f+\phi)\leq\frac{1}{2}\sqrt{2}$, $u(f+\phi)\leq\frac{1}{2}$.*

The proofs of these theorems will be given in Sections 5–8.

3. Proof of Theorem 1. If $f_1(z)=P(z)+Af(z)$, where $P(z)$ is any polynomial and A any nonzero constant, then for any number $c>0$,

$$M(r,f)/\mu(r,f)(\nu(r,f))^c\sim M(r,f_1)/\mu(r,f_1)(\nu(r,f_1))^c \tag{3.1}$$

and so we may suppose without loss of generality that

* For the definition *mean type of order ρ*, see, for example, R. P. Boas, Jr., Entire Functions, New York, 1954.

$$f(z) = \sum_1^\infty a_n z^n = \sum_1^\infty z^n \exp(i\alpha_n)/(R_1 \cdots R_n),$$

where $0 < R_1 < R_2 \cdots$. Let

$$S_n(\theta) = \sum_1^n \exp(ij\theta + i\alpha_j), \qquad S_0(\theta) = 0;$$

$$f_M(z) = f_M(re^{i\theta}) = \sum_1^M r^n \exp(in\theta + i\alpha_n)/(R_1 \cdots R_n)$$

$$= \sum_1^M S_n(\theta) \frac{r^n}{R_1 \cdots R_n}\left(1 - \frac{r}{R_{n+1}}\right) + \frac{r^{M+1}S_M}{R_1 \cdots R_{M+1}}.$$

Let $R_k \leqq r < R_{k+1}$, $n_0 < k < M-1$ and write

$$\sum_1 = \sum_1^{k-1} \frac{n^\alpha r^n}{R_1 \cdots R_n}\left(\frac{r - R_{n+1}}{R_{n+1}}\right), \qquad \sum_2 = \sum_k^M \frac{n^\alpha r^n}{R_1 \cdots R_n}\left(\frac{R_{n+1} - r}{R_{n+1}}\right).$$

Then $|f_M(z)| \leqq C(\sum_1 + \sum_2 + O(1/M))$, $\sum_1 < \mu(r)(\nu(r))^\alpha$,

$$\sum_2 \leqq \alpha \sum_{k+1}^M \frac{r^n n^{\alpha-1}}{R_1 \cdots R_n} + C_2 \sum_{k+1}^M \frac{r^n n^{\alpha-2}}{R_1 \cdots R_n} + \mu(r)\nu(r))^\alpha$$

$$< \{\alpha + C_2/\nu(r)\} \sum_3 + \mu(r)(\nu(r))^\alpha,$$

where C_2 is a constant and

$$\sum_3 < \left(\sum_{k+1}^P + \sum_{P+1}^\infty\right)\left(\frac{r^n n^{\alpha-1}}{R_1 \cdots R_n}\right) = \sum_4 + \sum_5, \text{ say.}$$

Take $P = [(k+1)/C_1]$. Then

$$\sum_4 < \mu(r)\{(C_1^{-\alpha} - 1)(k+1)^\alpha/\alpha + (k+1)^{\alpha-1}\},$$

$$\sum_5 \leqq (P+1)^{\alpha-1} \sum_{P+1}^\infty \mu(r)(r/R_{P+1})^{n-P}.$$

Now $R_{p+1} \geqq R_{k+1}\{(P+1)/P\}^\alpha$, $R_{k+1} > r$. Hence

$$\sum_5 < \{\mu(r)P^\alpha\}/\{(P+1) - (P+1)^{1-\alpha}P^\alpha\}.$$

Letting $M \to \infty$ we have

$$M(r,f) < C\mu(r,f)(\nu(r,f))^\alpha \left\{2 + (C_1^{-\alpha} - 1)\left(\frac{k+1}{k}\right)^\alpha + C_1^{-\alpha} + o(1)\right\}$$

and the theorem follows.

4. Corollary and example. If we take $C = 1 = \alpha$, condition (i) is satisfied and (1.5) follows from (1.4).

Example. Let

$$f(z) = \sum_1^\infty z^n \exp(in \log n)/(R_1 \cdots R_n),$$

where $R_n = (n+1)^a(\log(n+1))^{a_1}$, $a>0$; or $a=0$, $a_1>0$. The conditions (i)–(iii) are satisfied with $\alpha=\frac{1}{2}$ ([5], pp. 116–118), C an absolute constant, and C_1 any number <1. From (1.4) we have

$$\limsup_{r\to\infty} M(r,f)/\{\mu(r,f)(\nu(r,f))^{1/2}\} \leqq 3C.$$

5. Proof of Theorem 2. (i–a). Consider first

$$\phi(z) = \sum_1^\infty z^n/(R_1 \cdots R_n), \qquad 0 < R_1 < R_2 \cdots, \ R_n \sim R_{n+1}.$$

For $R_k \leqq r < R_{k+1}$ and p any fixed number,

$$\frac{M(r,\phi)}{\mu(r,\phi)} > \frac{r}{R_{k+1}} + \cdots + \frac{r^p}{R_{k+1}\cdots R_{k+p}} > \frac{1}{2}p$$

for $r>r_0(p)$. Hence $\liminf_{r\to\infty} M(r,\phi)/\mu(r,\phi) \geqq \frac{1}{2}p$. Since p can be chosen arbitrary large

$$\lim_{r\to\infty} M(r,\phi)/\mu(r,\phi) = \infty. \tag{5.1}$$

(i-b). Consider now

$$\phi_1(z) = \sum_1^\infty z^n \exp(i\theta_n)/(R_1 \cdots R_n),$$

where $0<R_1<R_2, \cdots, R_{n+1}\sim R_n$. Let

$$f_1(z) = \sum_1^\infty z^n/(R_1 \cdots R_n)^2, \qquad f_2(z) = \sum_1^\infty z^{2n}/(R_1 \cdots R_n)^2.$$

We have

$$\mu(r,f_2) = \{\mu(r,\phi_1)\}^2 = \mu(r^2,f_1),$$
$$\{M(r,\phi_1)\}^2 \geqq M(r,f_2) = M(r^2,f_1).$$

Hence by (i-a)

$$M(r,\phi_1)/\mu(r,\phi_1) \geqq \sqrt{\{M(r^2,f_1)/\mu(r^2,f_1)\}} \tag{5.2}$$

which tends to ∞ as $r\to\infty$.

(i-c). Any entire function $f(z)$ satisfying condition (ii) of Theorem 1 and (i) of Theorem 2 can be written as $f(z)=P(z)+A\phi_1(z)$, where P is a polynomial and A a constant. Hence from (3.1) and (5.2), (i) follows.

(ii-a). Consider first

$$\phi(z) = \sum_1^\infty z^n/(R_1 \cdots R_n), \quad 0 < R_1 < R_2 \cdots; \quad \limsup_{n\to\infty} R_{n+1}/R_n = L > 1.$$

Then $R_{n+1} > L_1 R_n$ where $1 < L_1 < L$ for a sequence of values of $n = n_p \uparrow \infty$. Write $n_p = N$ and consider $M(r, \phi)/\mu(r, \phi)$ when $r = \sqrt{L_1 R_N} = aR_N$ (say). Then $R_N < r < R_{N+1}$ and

$$\frac{M(r,\phi)}{\mu(r,\phi)} < \{1 + a^{-1} + a^{-2} + \cdots\} + \{aL_1^{-1} + a^2 L_1^{-2} + \cdots\} + o(1).$$

Hence

$$\liminf_{r\to\infty} M(r, \phi)/\mu(r, \phi) \leqq (\sqrt{L} + 1)/(\sqrt{L} - 1). \tag{5.3}$$

(ii-b). If

$$\phi_1(z) = \sum_1^\infty z^n \exp(i\theta_n)/(R_1 \cdots R_n), \quad 0 < R_1 < R_2 \cdots; \quad \limsup R_{n+1}/R_n = L,$$

then $M(r, \phi_1) \leqq M(r, \phi)$, $\mu(r, \phi_1) = \mu(r, \phi)$; and we can complete the argument as in (i-c).

(iii) We consider

$$\phi(z) = \sum_1^\infty z^n (R_1 \cdots R_n), \quad 0 < R_1 < R_2 \cdots; \quad \liminf_{n\to\infty} R_{n+1}/R_n = l > 1;$$

and then complete the argument as in (ii-b). We have $R_{n+1} > l_1 R_n$, where $1 < l_1 < l$ for $n > n_0$. Further, for $R_n \leqq r < R_{n+1}$,

$$\frac{M(r,\phi)}{\mu(r,\phi)} = 1 + \left(\frac{R_n}{r} + \frac{R_n R_{n+1}}{r^2} + \cdots\right) + \left(\frac{r}{R_{n+1}} + \frac{r^2}{R_{n+1}R_{n+2}} + \cdots\right)$$

$$\leqq 1 + \left(\frac{R_n}{r} + \frac{r}{R_{n+1}}\right) + 2(l_1^{-1} + l_1^{-1-2} + l_1^{-1-2-3} + \cdots) + o(1)$$

$$< 2 + 3l_1^{-1} + \{2/(l_1^3 - 1)\} + o(1).$$

Hence

$$\limsup_{r\to\infty} M(r, \phi)/\mu(r, \phi) \leqq 2 + 3l^{-1} + \{2/(l^3 - 1)\}.$$

(iv) From the argument for (iii), we have $u(f) \geqq \frac{1}{2}$. If

$$\phi_1(z) = \sum_1^\infty z^n \exp(i\theta_n)/(R_1 \cdots R_n), \quad 0 < R_1 < R_2 \cdots; \quad \lim_{n\to\infty} R_{n+1}/R_n = \infty,$$

then

$$\phi_1\{R_n \exp(i(\theta_{n-1}-\theta_n))\} = 2\mu(R_n)\exp\{in\theta_{n-1} - i(n-1)\theta_n\} + o(\mu(R_n)).$$

Hence $u(\phi_1) \leqq \frac{1}{2}$ and (iv) follows by an argument similar to (i-c).

(v) By the argument of (ii) we have $U(f) \geqq 1$, and hence (v) follows from (1.1).

6. Proof of Theorem 3. Write

$$f(z) = \sum_0^\infty a_n z^n, \qquad G(z) = \sum_0^\infty b_n z^n, \qquad F(z) = \sum_0^\infty c_n z^n.$$

Then

$$\begin{aligned} a_n &= c_n b_0 + c_{n-1}b_1 + \cdots + c_0 b_n \\ &= \frac{1}{2\pi i}\int_\Gamma G(z)\left(\frac{c_n}{z} + \cdots + \frac{c_0}{z^{n+1}}\right)dz, \end{aligned} \tag{6.1}$$

where Γ is $|z| = R$. Hence

$$|a_n| r^n \leqq M(R, G)\left\{r^n|c_n| + r^{n-1}|c_{n-1}|\left(\frac{r}{R}\right) + \cdots + |c_0|\left(\frac{r}{R}\right)^n\right\}.$$

Let $R > r$. Then

$$|a_n| r^n \leqq M(R, G)\mu(r, F)\left\{\sum_0^\infty \left(\frac{r}{R}\right)^n\right\} = M(R, G)\mu(r, F)\{R/(R-r)\}.$$

Hence

$$\mu(r, f) \leqq M(R, G)\mu(r, F)\{R/(R-r)\}. \tag{6.2}$$

Taking $R = ar$, $a > 1$, and assuming (i)–(iii), we have

$$\frac{\mu(r, f)}{M(r, f)} \leqq O\left(\frac{M(ar, G)\mu(r, F)}{M(r, G)M(r, F)}\right) = o(1), \tag{6.3}$$

and so $U(f) = 0$. If one of the three conditions, say (iii), holds for a sequence of values of $r \uparrow \infty$, and the remaining two, (i) and (ii), for r tending to infinity, then from (6.3) we have $u(f) = 0$.

7. Remarks. (i). If we take $F(z) = \exp(Az^p)$, A any constant, p a positive integer, then condition (ii) of Theorem 3 is satisfied with $\alpha = \frac{1}{2}p$ [3].

(ii) If $f(z) = \exp(Az^p)P(z)$, P any polynomial, p any positive integer, then $U(f) = 0$.

(iii) If

$$F(z) = \sum_0^\infty c_n z^n, \qquad G(z) = \sum_0^\infty b_n z^n, \qquad f(z) = F(z)G(z)$$

satisfy the following:

(a) $$\sum_0^\infty |b_n| r^n = O(M(r, G)), \qquad \text{as } r \to \infty;$$

(b) $$M(r, F)M(r, G) = O(M(r, f)), \qquad \text{as } r \to \infty;$$

(c) $$U(F) = 0,$$

then $U(f) = 0$. For from (6.1) we have

$$r^n |a_n| \leqq |c_n| r^n |b_0| + \cdots + |c_0| r^n |b_n|$$

$$\leqq \mu(r, F) \left\{ \sum_0^\infty |b_n| r^n \right\} \leqq O(\mu(r, F)M(r, G)).$$

Hence $\mu(r, f) \leqq O(\mu(r, F)M(r, G))$, and so from (b), (c), we have $U(f) = 0$. In particular, if b_n and c_n be nonnegative for $n > n_0$, and $U(F) = 0$, then $U(f) = 0$.

Example. Let

$$f(z) = F(z)G(z), \qquad F(z) = e^{-z}, \qquad G(z) = \sum_0^\infty (-1)^n |b_n| z^n,$$

where b_n are any numbers such that $G(z)$ is an entire function. Then $M(r, f) = M(r, F)M(r, G)$, $M(r, G) = \sum_0^\infty |b_n| r^n$, $U(F) = 0$. Hence $U(f) = 0$.

8. Proof of Theorem 4. We have

$$\mu(r, f + \phi)/M(r, f + \phi) \leqq \{1 + o(1)\}\mu(r, f)/M(r, f).$$

For $r > r_0(\delta)$,

$$\{\mu(r, f)\}^2 < I(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \frac{\pi + 2\rho\delta}{2\pi} \{M(r, f)\}^2 + O(1),$$

and, for a sequence of values of $r \uparrow \infty$, $2\{\mu(r, f)\}^2 < I(r)$, and the theorem follows.

Remark. If $f(z) = \sum_0^\infty a_n z^n$ be any entire function such that $M(r, f) \sim \sum_0^\infty |a_n| r^n$, then $u(f) \leqq \frac{1}{2}$. *Added in proof:* If $P(z)$ is any polynomial of degree n, then it can be proved, by an argument similar to that of W. K. Hayman (J. Reine Angew. Math., vol. 196, 1956, pp. 67–95), that $\mu(r, e^P)/M(r, e^P) = O(r^{-n/2})$.

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A PROPOSAL OF MARRIAGE*

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1. Introduction. A great deal of effort is currently being devoted to revising the school and college mathematics curricula. New material and new points of view are being introduced through the organizing of special courses, the development of new textbooks, the writing of articles, and so on.

The present period is an opportune one for this sort of activity. There has for some time been a need for change, and, moreover, most of us in this post-sputnik era are ready to change or even anxious to change.

There need be no quarrel about the desirability of the goals being sought. Most of us would agree, for example, that it is desirable for students to understand the concept of a set or the notion of a function as early as possible, and that it is desirable to appreciate abstractness and axiomatics. One could perhaps argue about exactly *when*, or exactly *how*, these ideas should be introduced. But the question being raised here concerns the possibility that, in our enthusiasm for the new approach, we are neglecting to develop still another equally important aspect of the needed change.

My answer to this question is that we are in fact neglecting one very important point of view, and one which could easily be wedded to what is presently being adopted. In the next section I describe what I believe is being neglected. Then in the following section I indicate a number of ways in which I think the situation could be improved. Finally in the remaining sections of the paper I give a precise treatment of one example, to show explicitly how the idea I am proposing would work out in practice. The example will be concerned with ordinary differential equations. Our treatment will contain some novel features, but we shall be primarily concerned with the *way* in which they are treated.

2. What is being neglected? The neglected point of view to which I refer is that of the applied mathematician. And it is my contention that his point of view is not only compatible with that of the pure mathematician, but that the two cannot live without each other. I am therefore proposing a marriage of these two points of view.

That the principals need each other follows from the way in which mathematics has developed in the past. They certainly appear together in the work of such very great mathematicians as Archimedes, Newton, Euler, Gauss, Poincaré, and von Neumann. Much of the initial motivation and stimulation for past research in pure mathematics has come from the problems of applied mathematics. On the other hand the applied mathematician looks to the pure mathematician for generalizations and refinements of his techniques, and sometimes even for whole new techniques.

When the two mathematicians are not speaking to each other the cause of

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each will suffer. Consider the long delay in developing the Laplace transform, which was caused by Heaviside and what he called "those Cambridge mathematicians" being so disdainful of each other.

Mathematical activity is now increasing at a very rapid rate. This means it is most important that we understand the various aspects of this growth. But it also means that we are in great danger of specializing and thereby losing just the understanding that we need. This danger is intensified for the student who is under so much pressure to complete his thesis, or to pass a particular course. It can also be intensified later by university administrations if they insist on the appearance of a large number of publications in the journals.

Such specializing leads, on the one extreme, to the pure mathematician who is concentrating on just one small point in some abstract point set. He might even publish several papers on this one point. On the other extreme, we have the applied mathematician who is concentrating on the intricacies of a large, inelegant, inefficient, and probably unnecessary computer program. He scorns, and is scorned by, his pure colleague.

It is my claim that the points of view being caricatured by these two extremes belong together. Both the rigor and the realism of mathematics need each other. Alone they can only be sterile.

3. What can be done? To bring these points of view together quickly and effectively, we must do so in the classroom. What we are here concerned about is the manner in which material is presented. The two points of view are rarely presented together in a proper way. Pure mathematicians seem to be in a majority in university mathematics departments, and so it turns out that it is the point of view of the applied mathematician that is most often neglected. This is especially unfortunate for the students, since most of them are primarily interested in applied mathematics.

I am not advocating that we set up separate pure and applied courses, so that the student will have to choose between them. I am claiming instead that if a course in mathematics is taught properly it will thereby necessarily contain the essential ingredients of both points of view. In keeping with our allusions to the possibility of marriage, this ideal is perhaps a case of "two living as cheaply as one."

There are many ways in which this ideal can be approached. I would like to illustrate what I mean with a few examples, appropriate to courses usually taught at the intermediate college level. Because I feel that it is the applied aspect which is most often neglected, my examples will mostly take the form of suggestions for modifying presently existing courses so that they will include enough of the motivation and understanding that comes from a proper consideration of the applications.

Let us begin by considering the introduction of the definite integral. The definition and the fundamental theorem are basic to any theoretical considerations. Numerical quadrature is on the other hand one of the basic problems in

practice. These ideas are conceptually very close together, and they can be taught together quite naturally. The student can in this way be led to a more thorough understanding of both the meaning of a definite integral, and of the problem of integration in practice.

The situation referred to in the preceding paragraph is typical. The theory is concerned with a particular limit, while the practice is concerned with a finite approximation, and hence with the consequences of stopping just short of this limit. The connection between these two points of view is perhaps even more dramatically illustrated in the case of ordinary differential equations. Beginning with the next section I shall show how easy and how profitable it is to consider such equations from both points of view at the same time.

As for differentiation, an early application could be to the use of Newton's method for finding the zeros of a polynomial. For a complete understanding of the method one could write a computer program, for which some simple interpretive language should be available. A useful application of partial differentiation which is usually omitted is in finding the normal equations which result when a linear expression is fitted to data by "least-squares." And this brings up the subject of matrices.

In courses concerned with matrices there are almost unlimited opportunities to at least refer to the interesting, practical, and nontrivial problems related to matrix inversion and finding eigenvalues.

Applications can be introduced in many other situations as well. There are of course the standard applications of calculus and differential equations in physics, of complex variables in circuit theory, and of mathematical statistics in the biological and social sciences. Linear programming and the simplex method can easily be introduced in a course on finite-dimensional spaces.

In the examples referred to above I have emphasized especially the numerical aspects. This is partly because the basic numerical problem happens to be very close to the basic theoretical problem in so many branches of mathematics (particularly in analysis, of both the hard and the soft varieties). It is also partly because I believe that an interplay between pure and applied is most urgently needed in this area. Moreover numerical analysis is the one area of applied mathematics which is of common interest to all users of mathematics.

We can blame the computers for opening up vast new problem possibilities in this area. (But we can also thank them for taking out of numerical analysis most of the drudgery of making calculations.) Just one such possibility comes from looking upon mathematics as an experimental science. In the past, pencil and paper were the most common pieces of apparatus. Now with a computer it is possible to perform experiments on a scale and of a kind which were quite unthinkable to most of us only ten years ago. A great deal of thought must be put into the design and interpretation of these experiments. And to be profitable, much of this thought will have to be motivated and encouraged by a joining together of the point of view of the mathematician who has everyday problems

to solve, with that of the mathematician who can afford to remain relatively detached and uncommitted.

4. An example concerning numerical methods and existence theorems. I shall now present an example of the sort of outcome we should expect from the proposed marriage. The example is an outline of the basic steps to be used in a general treatment of ordinary differential equations. It is in my opinion a good example of how one can teach both the applied and the pure aspects of a subject at the same time, and to thereby gain a deeper understanding of each one separately. In this case the applied aspect is represented by numerical methods of solution, while the pure aspect is represented by existence theorems.

I also claim that the numerical methods are introduced in a new and very efficient way, and that the existence proof has been improved somewhat.

The problem to be considered concerns the solution (if any) of the following:

$$x' = f(t, x), \qquad x(t_0) = x_0,$$

where for the moment we assume only continuity of $f(t, x)$ in some region of the tx-plane which contains the point (t_0, x_0).

A treatment of this first-order problem is basic from the point of view of both the numerical methods and the existence theorems. This is so because a system of first-order problems can be handled in almost exactly the same way, from both points of view, by considering x and f to be vectors and considering absolute values to be norms. And of course any nth-order problem can be interpreted as a system of n first-order problems.

Again from both points of view, it is natural to begin consideration of the problem with some approximation to the hoped-for solution. We can define such an approximation by first introducing the numbers t_i with $t_i - t_{i-1} = h$ for $i = 1, 2, \cdots$. We then by some means or other obtain some numbers y_i which are meant to approximate the values $x_i = x(t_i)$, if such exist. Our approximate solution is then obtained by linear interpolation between succeeding points (t_i, y_i) and will be denoted by $y(t)$.

What is crucial is, of course, the way in which the y_i are defined. Let us suppose that $y_0, y_1, \cdots, y_{n-1}$ have already been defined and that we require a formula to define y_n. To motivate the development of suitable formulas we first point out that, if there is a solution $x(t)$ of the original problem, it will, by the mean-value theorem, have to satisfy

$$x_n = x_{n-1} + hx'(\xi) = x_{n-1} + hf(\xi, x(\xi)),$$

where $(\xi, x(\xi))$ is some point in the strip of the tx-plane between the abscissas t_{n-1} and t_n. We of course do not know ξ, but for our approximation we are led in a natural way to take $y_0 = x_0$, and to try

$$y_n = y_{n-1} + h\bar{f}_n, \tag{1}$$

where $\bar{f}_n$ is the average of some values of f in the neighborhood of the last com-

puted point (t_{n-1}, y_{n-1}).

Here and in what follows it is useful to have in mind the standard geometrical interpretation in terms of the direction field. It also justifies the use of terms like "neighborhood" and "point."

The simplest possible choice for $\bar{f}_n$ is $f(t_{n-1}, y_{n-1})$, and this choice defines Euler's method. It is good enough for the proof of the basic existence theorem, but too crude for numerical work. With the more sophisticated formulas of this section, we shall see that we can obtain well-known and useful numerical methods without in any way increasing the complexity of the existence proof. We thereby also establish the convergence of these methods.

There are two principal ways in which $\bar{f}_n$ is defined in numerical work. One way leads to Runge-Kutta methods. These methods are obtained by expressing $\bar{f}_n$ as a linear combination of k values of f,

$$\bar{f}_n = c_1 f_{n,1} + \cdots + c_k f_{n,k},$$

where $f_{n,1} = f(t_{n-1}, y_{n-1})$, and where the other values of f are obtained in turn from formulas of the form

$$f_{n,2} = f(t_{n-1} + \alpha h, y_{n-1} + \alpha h f_{n,1}),$$
$$f_{n,3} = f(t_{n-1} + \beta h + \gamma h, y_{n-1} + \beta h f_{n,1} + \gamma h f_{n,2}),$$

and so on. (These formulas may appear at first to be awkward, but they have a completely natural and simple geometrical interpretation. For example, try the case with $k=3$, $c_1=\frac{1}{6}$, $c_2=\frac{4}{6}$, $c_3=\frac{1}{6}$, $\alpha=\frac{1}{2}$, $\beta=-1$, $\gamma=2$.) In practice k is usually taken to be 4. The remaining parameters are determined primarily by equating coefficients of like terms on either side of (1), when the two sides of (1) are replaced by expansions in powers of h about the point (t_{n-1}, y_{n-1}). (The expansions are those obtained by substituting a solution $x(t)$ for $y(t)$ into each side of (1) and assuming sufficient differentiability of x and f.) For our present purposes it is enough to note that the first equation so obtained simply requires that $\sum_{i=1}^{k} c_i = 1$. This condition was to be expected since $\bar{f}_n$ was to be an "average" value of f. The condition is the special case here of what we later call "consistency." It is enough for our present purposes because we can prove existence theorems with the approximate solutions generated by *any* consistent Runge-Kutta method.

The other principal way of defining $\bar{f}_n$ leads to Adams's methods. They are obtained by expressing $\bar{f}_n$ as a linear combination of values of f at the points (t_i, y_i). Putting $y_i' = f(t_i, y_i)$ we write

$$\bar{f}_n = b_k y'_{n-k} + b_{k-1} y'_{n-k+1} + \cdots + b_0 y'_n.$$

The b's are found as before by equating coefficients in (1) after expanding in powers of h, but the calculations are now quite a bit simpler. We note that the first equation again gives us the consistency condition $\sum_{i=0}^{k} b_i = 1$.

Before proceeding to the existence theorem we should point out that some

new difficulties have arisen. We notice that some other procedure will be needed to obtain the starting values $y_1, \cdots, y_{k-1}$. And we also notice that (1) now defines y_n only implicitly if $b_0 \neq 0$. Another formula will then be needed for predicting a trial value of y_n, before we can iterate on (1). On the other hand fewer evaluations of f will be needed at each step with Adams's methods, and this could make them worth the extra trouble compared to Runge-Kutta methods.

Let us turn now to the question of existence. We want to show that any consistent Runge-Kutta or Adams's method can be used as easily as Euler's method in the proof of existence. At the same time we want to show that the usual existence proof can be improved somewhat. If we were to assume a Lipschitz condition at this stage we would be able to avoid the use of equicontinuity and Ascoli's lemma. We prefer however to present the more general result that continuity *alone* implies existence, a Lipschitz condition being needed only for uniqueness.

One introduces a region in the tx-plane, and an appropriate interval of the t-axis, in the usual way. Then for t, t' in this interval, one obtains (directly from the definition of $y(t)$, that is, from (1) followed by linear interpolation) the following inequality

$$| y(t) - y(t') | \leqq M | t - t' | ,$$

where M is the usual bound for $|f(t, x)|$ times $\sum |b_i|$ or $\sum |c_i|$.

Since this result is independent of h we conclude that the approximations for different values of h are equicontinuous as well as being uniformly bounded. Ascoli's lemma therefore guarantees that some subsequence of these approximations will converge, as $h \to 0$, uniformly to a limit function which we denote by $Y(t)$. The limit is continuous because the approximations are continuous.

To show that this limit is a solution of the original problem we choose a typical member $y(t)$ of the convergent subsequence and write

$$y(t) - x_0 = (y(t) - y_n) + (y_n - y_0),$$

where $n = [t/h]$. Then

$$y(t) = x_0 + (t - t_n) f_n + \sum_{i=0}^{n-1} h \bar{f}_i .$$

The uniform convergence of $y(t)$ to $Y(t)$, and the continuity of $f(t, x)$, guarantee that the sum on the right approaches a limit as $h \to 0$ so that we end up with

$$Y(t) = x_0 + \int_{t_0}^{t} f(u, Y(u)) du,$$

and we have thus completed the existence proof.

Besides having our subsequence of approximate solutions approaching the limit $Y(t)$, we also have their slopes, where they exist, approaching the limit $Y'(t)$. The existence proof is improved to the extent that this result is not con-

sidered to be an integral part of the proof, as it usually is, but only as a corollary. As a matter of fact this result does not even hold for some of the more general approximate solutions to be considered in the next section.

Finally we point out that uniqueness can be established in the usual way, once some sort of Lipschitz condition has been assumed. It is here that one might introduce the method of successive approximations and refer to Picard's form of the existence and uniqueness theorem. But for our purposes this form of the theorem is of relatively little interest.

5. The general case. Let us now consider very briefly the general multistep methods. Adams's methods are special cases of these general methods, but the latter will have one important feature not found in either Runge-Kutta or Adams's methods. This feature makes possible the phenomenon of instability, which in turn leads to questions which are important both for numerical calculations and for existence theorems.

The general multistep methods can be introduced in a natural way as follows. We recall that in Adams's methods the use of earlier values of y_i' caused trouble with getting the procedure started. If we are however willing to put up with this extra trouble, we might as well also make use of the information we have about the earlier values of y_i. This leads us to consider

$$y_n = a_k y_{n-k} + \cdots + a_1 y_{n-1} + h(b_k y'_{n-k} + \cdots + b_0 y'_n) \tag{2}$$

in place of (1). Having almost twice as many parameters as (1), we might expect that it will be possible to make such a formula more accurate than (1). This will turn out to be the case, but not, as we shall see, in the way we might at first expect.

Of course we will have to have $\sum_{i=1}^{k} a_i = 1$, and this turns out to be the first equation we get from (2) on expanding in powers of h and equating coefficients. But we do not this time expect the sum of the b's to be unity, because we are no longer extrapolating from the value y_{n-1} at t_{n-1}. We are instead extrapolating from an average value of earlier y's, at the value of t at which this average is effective. Taking moments about t_n we find that this value of t, say t^*, must satisfy

$$t_n - t^* = ha_1 + 2ha_2 + \cdots + kha_k.$$

We are thus led to the equation $\sum_{i=0}^{k} b_i = \sum_{i=1}^{k} ia_i$, and this turns out to be the second equation obtained from (2) by expanding and equating coefficients. We now have two equations to be satisfied by the a's and b's. These two equations together guarantee the "consistency" of the methods.

The new feature that appears with (2) can be described in the following way. We notice that both of the difference equations (1) and (2) will in general have more than one solution. Since only one of their solutions will be close to the solution of the differential equation for small h, there is the danger that it will be dominated by one of the extraneous solutions of the difference equation.

Then both the numerical calculation and the proof of existence would be invalidated. Such unwanted behavior of the extraneous solutions is called instability. The new feature is that with (2) it is possible to have such instability, whereas with (1) the extraneous solutions all approach zero as $h \to 0$.

In trying to make (2) as accurate as possible we would naturally try to equate as many coefficients as possible in the expansions of the two sides of (2). It turns out that we cannot in general match up any more coefficients in (2) than we can for Adams's formula, with the same k, if we at the same time insist on stability. Matching more coefficients would give us formulas which represent the differential equation more accurately, but such formulas are too sensitive to be stable.

I shall not go further into details but I should like to indicate two parallel directions in which we could go from here. One direction we could go is towards the development of efficient numerical methods. It is natural of course to begin with the consistent and stable methods, and to ask which from all these methods is best in practice. The additional degrees of freedom afforded by the multistep methods can be used to minimize the error which accumulates during a long sequence of calculations. This point of view has been exploited by Hull and Newbery [7] who are in this way led to a number of new methods which are in a certain sense best possible.

The other direction in which we might go concerns the establishing of existence theorems and so also the convergence of the multistep methods. Such questions turn out to be much more difficult when one begins with the approximate solutions which are generated by multistep methods. It turns out however that existence and uniqueness can be established if and only if the multistep method is both consistent and stable. The meaning of "stable" must of course be made more precise, and there are some surprises such as the fact that the derivatives do not in general converge. The details are given by Dahlquist [2, 3], and by Hull and Luxemburg [6].

6. Concluding remarks. In the last two sections I have summarized very briefly a treatment of ordinary differential equations which combines the point of view of the pure mathematician interested in existence theorems with that of the applied mathematician interested in numerical calculations. A great deal has of course been omitted. Coddington and Levinson ([1], Chs. 1, 2) give a complete treatment of questions regarding the weakening of hypotheses, the continuing of solutions, the dependence on parameters, and so on. Hildebrand ([5], Ch. 6) and Henrici [4] give a complete treatment of questions regarding the rate of convergence, convenience in practice, round-off, and so on.

One of the chief advantages of the presentation given in this paper is that the numerical methods have been introduced in a way which is both completely natural and completely general. All too often only very special cases are treated, and they are usually derived from a bewildering variety of formulas for numerical differentiation and numerical integration, or from a juggling around with

finite difference formulas. Such formulas have their uses of course, and so even do the unstable methods they sometimes yield, but they ought not to be mixed up in a discussion of differential equations.

An advantage of using realistic numerical procedures in the proof of an existence theorem is that we thereby at the same time establish convergence of the numerical procedures. In the case of multistep methods it is particularly important to know which methods converge and which do not.

The early sections of this paper were devoted to advocating a joining together of the pure and applied points of view in mathematics. Several areas of mathematics were referred to in which this joining together ought to be considered. In the last two sections I have shown, by way of illustration, just how advantageous such a union can be in one particular situation.

Of course the advantages will vary tremendously in kind and degree between different areas of mathematics, and between different applications. But I believe there are worthwhile advantages in each of the situations already referred to.

I therefore wish to return to and repeat my original proposal of a marriage between the pure and applied points of view. I hope they will learn to live together, happily ever after.

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ON THE AREA OF CURVED SURFACES

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It is a common impression that if one is to say anything rigorously correct about the area of a surface he must first spend several months of his life with some of the most difficult books in the mathematical library. Be that as it may, certainly there are no textbooks of calculus, or of advanced calculus, which present more than a heuristic account of the subject even for nonparametric surfaces. It would seem worthwhile, therefore, to present a simple account of the basic theory of surface area, and it is to this end that the paper is devoted.

Starting from Lebesgue's definition of area, we shall show that the fundamental results of the theory (for surfaces of the form $z=f(x, y)$) follow at once from a single inequality whose geometric intent is roughly that integral smoothing is an area shrinking operation. The inequality itself is due to Radó, but our proof is a considerable simplification of his, and at the same time gives the result in a somewhat more general form. It is hoped that the treatment will be within the ability of an honors class in advanced calculus. If so, then the theory of surface area can take a place alongside arc length as a possible part of the curriculum.

1. Definition of surface area. In this section we shall briefly outline Lebesgue's concept of surface area. As motivation for this, it is convenient to begin by considering the class of polyhedral surfaces. Here the notion of area is entirely elementary; if $z=f(x, y)$ is such a surface, then we have

$$\text{Area} = \sum \text{areas of the plane faces} = \int_R \sqrt{(1 + f_x^2 + f_y^2)}\, dA,$$

the integral being evaluated over the set R where the surface is defined. We assume throughout that R is a closed region whose boundary is a rectifiable curve of finite length.

To define the area of a *curved* surface, one might simply decide to use the same integral expression. This procedure, however, tends to blur the geometrical idea of area, and in any case is analytically vague: To what class of functions should the formula be applied? Should one allow Lebesgue integration, or must the function $\sqrt{(1+f_x^2+f_y^2)}$ be strictly Riemann integrable? etc. That these questions are relevant is clear if we consider a surface $z=g(x)$ for which g is a monotonically increasing function whose derivative is zero almost everywhere. If R is a unit square, we arrive at the absurd conclusion that the area of the surface is one.

A more appealing definition parallels that of arc length; that is, the area of the surface is to be the limit of the areas of approximating polyhedra. Unfortunately, we have only to recall Schwarz's famous example [1] to see the incorrectness of this method.

Lebesgue's happy idea was to modify the preceding definition by using the

lower limit of the areas of approximating polyhedra. To state this precisely, let us introduce for piecewise smooth functions $u(x, y)$ the notation

$$I[u] = \int_R \sqrt{(1 + u_x^2 + u_y^2)}\, dA$$

(the area of a polyhedral surface $z=f(x, y)$ over R is thus simply the integral $I[f]$). Now let $f(x, y)$ be a real continuous function defined on R, and let $\{f_n(x, y)\}$ be a sequence of quasilinear functions such that $f_n(x, y) \to f(x, y)$ uniformly on R. We set $A = \liminf I[f_n]$, observing that A depends on the particular sequence of functions f_n.

The Lebesgue area $L[f]$ of the surface $z=f(x, y)$ is then defined to be the infimum of the numbers A corresponding to all possible sequences $\{f_n\}$ converging to f as above.

This definition may alternatively be expressed in the convenient and concise form

$$L[f] = \operatorname*{minimum\ limit}_{f_n \to f} I[f_n].$$

It should be emphasized that this definition gives a value (possibly infinite) for the area of *any* continuous surface over R. Moreover, the area thus defined is clearly a lower semicontinuous functional with respect to uniform convergence, that is, if $f_n \to f$ uniformly on R, then $L[f] \leqq \liminf L[f_n]$. The property of lower semicontinuity is of considerable importance in the general theory of area, though we shall make no particular use of it here.

In order that the Lebesgue's definition of surface area be useful, it must still be verified that the Lebesgue area of a polyhedron agrees with its elementary area. This follows, however, from our main proposition, which we now state.

THEOREM 1. *If f is continuously differentiable, then $L[f] = I[f]$.*

Theorem 1 obviously justifies the well-known formula for surface area. The proof of Theorem 1 will be given in the next three sections of the paper.

2. Preliminary results. We shall be interested in the operation of integral averaging or smoothing. Let us associate with each positive number h a corresponding function $K(\xi, \eta) = K(\xi, \eta; h)$ with the following properties:

(i) K is nonnegative and continuously differentiable for all values of ξ and η,
(ii) $K \equiv 0$ outside the circle $\xi^2 + \eta^2 = h^2$,
(iii) $\int K(\xi, \eta)\, dA_{\xi\eta} = 1$.

In (iii) and in subsequent formulas it is tactitly assumed that the integration is carried out over all values of ξ and η.

The *integral average* ϕ_h of a function $\phi(x, y)$ is now defined by the formula

$$\phi_h = \phi_h(x, y) = \int K(\xi, \eta)\phi(x + \xi, y + \eta)\, dA_{\xi\eta}, \tag{1}$$

or, alternatively, by the equivalent expression

$$\phi_h = \int K(\xi - x, \eta - y)\phi(\xi, \eta)dA_{\xi\eta}. \tag{2}$$

Certain properties of the integral average will be needed later on. Assuming to begin with that ϕ is continuous, then by virtue of the properties of $K(\xi, \eta)$ one has

$$\phi_h(x, y) \to \phi(x, y) \text{ as } h \to 0. \tag{3}$$

Moreover, from (2) it follows that ϕ_h is continuously differentiable; indeed, we have

$$\phi_{hx} = -\int_R K_\xi(\xi - x, \eta - y)\phi(\xi, \eta)dA_{\xi\eta}, \tag{4}$$

and a similar formula holds for ϕ_{hy}. Finally, if ϕ is piecewise smooth, differentiation of (1) yields

$$\phi_{hx} = \phi_{xh}, \qquad \phi_{hy} = \phi_{yh}. \tag{5}$$

One more elementary fact will be required later:

If $F(p, q)$ is a convex function of the variables p and q, and if u and v are functions of (x, y), then $F(u_h, v_h) \leqq F(u, v)_h$.

This is nothing more than Jensen's inequality. (See, e.g., G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, 1925, p. 53.)

3. The fundamental inequality. Let $f(x, y)$ be a continuous function defined in R. Its integral average f_h is defined in the subregion R_h of R consisting of all points of R which are farther than h from the boundary. We put

$$I[f_h] = \int_{R_h} \sqrt{(1 + f_{hx}^2 + f_{hy}^2)}dA,$$

where the fact that the integration is only over the subregion R_h should cause no confusion. Then for all $h > 0$ we have

$$I[f_h] < L[f]. \tag{6}$$

This is the fundamental inequality (*cf.* also the last paragraph of Sec. 4).

In order to prove (6), let $\{f_n\}$ be a sequence of quasilinear functions with the property that

$$\lim f_n = f, \qquad \lim I[f_n] = L[f]; \tag{7}$$

such a sequence surely exists in view of the definition of area. Now (5) and (4) imply

$$|f_{hx} - (f_n)_{xh}| = |f_{hx} - (f_n)_{hx}| \leqq C(h) \max |f - f_n|,$$

and a similar estimate holds with x replaced by y. Therefore, for any convex

function $F(p, q)$,

$$F(f_{hx}, f_{hy}) \leqq F((f_n)_{xh}, (f_n)_{yh}) + \epsilon \leqq F(f_{nx}, f_{ny})_h + \epsilon,$$

where $\epsilon = \epsilon(n, h) \to 0$ as $n \to \infty$. Supposing in particular that $F = \sqrt{(1+p^2+q^2)}$, integration of the preceding inequality over the region R_h yields

$$I[f_h] \leqq \int_{R_h} F(f_{nx}, f_{ny})_h dA + \epsilon \text{ meas } R_h.$$

Now the operation of averaging is in fact an integration, so that the first term on the right can be treated as an iterated integral. By reversing the order of integration we can write this term in the form

$$\int K(\xi, \eta) \int_{R'_h} F(f_{nx}, f_{ny}) dA_{xy} dA_{\xi\eta},$$

where R'_h denotes the parallel translation of the set R_h by the vector (ξ, η). The inner integral is clearly dominated by $I[f_n] - \text{meas } (R - R_h)$; whence by property (iii) of the function K we find

$$I[f_h] \leqq I[f_n] - \text{meas } (R - R_h) + \epsilon \text{ meas } R_h$$

Letting $n \to \infty$, and taking account of (7), leads at once to the required inequality.

4. Proof of Theorem 1. Since by hypothesis the partial derivatives of f are continuous, we have by (3), (5), and (6),

$$I[f] = \lim_{h \to 0} I[f_h] \leqq L[f].$$

On the other hand, there certainly exists a sequence of quasilinear functions tending to f such that $\lim I[f_n] = I[f]$, (the reader may verify that this can be attained even for inscribed polyhedra). Therefore from the definition of $L[f]$ it follows that

$$L[f] \leqq \lim I[f_n] = I[f].$$

Combining the two preceding inequalities yields $L[f] \leqq I[f] \leqq L[f]$, and the proof of Theorem 1 is complete.

Remark. By virtue of Theorem 1 it is evident that $I[f_h] = L[f_h, R_h]$, the notation being obvious. Hence the fundamental inequality can be written in the form

$$L[f_h, R_h] < L[f, R], \tag{6'}$$

and in this sense averaging is indeed an area shrinking operation.

5. Further results. In this section we shall consider some further properties of the Lebesgue area. Though these results do not have the fundamental im-

portance of Theorem 1, they nevertheless serve to round out the theory.

To begin with, we note from the introductory discussion that there are functions for which the integral $I[f]$ does not represent surface area. This fact leads to two main problems: first, to characterize precisely the class of functions for which $L[f]=I[f]$ (so far we know only that this class includes all continuously differentiable functions), and second, to give a constructive method for calculating the area of an arbitrary surface. The first problem is unfortunately beyond the scope of this paper, for a satisfactory answer turns out to require rather deep tools of the theory of integration [4]. On the other hand, the second problem has an elegant and simple solution.

Let us say that an averaging procedure is *regular* if there is a constant M, independent of h, such that

$$\int |\nabla K| \, dA_{\xi\eta} \leqq Mh^{-1}.$$

This condition is satisfied, for example, by any averaging procedure whose kernel is of the form $K(\xi, \eta)=h^{-2}\bar{K}(\xi h^{-1}, \eta h^{-1})$, where the function $\bar{K}(u, v)$ is independent of h. We now have the following

THEOREM 2. *For an arbitrary continuous function $f(x, y)$ defined on R, and for any regular averaging procedure, we have $L[f]=\lim_{h\to 0} I[f_h]$.*

Proof. Because of (6), it is obvious that $\lim I[f_h] \leqq L[f]$. To prove the opposite inequality, let us first extend f so that it is defined and uniformly continuous in some open region containing R. Denoting the resulting function by $g(x, y)$, we have $g \equiv f$ in R, while (for sufficiently small h) the integral average g_h exists and is continuously differentiable in all of R. Now for each $h>0$, let g_h' be a quasilinear function in R such that

$$|g_h' - g_h| \leqq h^{-1}, \qquad |I[g_h'] - I[g_h]| \leqq h^{-1}.$$

Obviously $g_h' \to f$ as $h\to 0$. Therefore by the definition of area,

$$L[f] \leqq \lim I[g_h'] = \lim I[g_h].$$

To complete the proof it is sufficient to show that the difference $I[g_h] - I[f_h] \to 0$ as $h\to 0$. By writing out the integrals involved, one easily verifies that

$$0 \leqq I[g_h] - I[f_h] \leqq \max\{1 + |g_{hx}| + |g_{hy}|\} \operatorname{meas}(R - R_h).$$

Now the boundary of R is rectifiable, so that the set $R-R_h$ can be covered by at most Lh^{-1} circles of radius h, where L depends only on the length of the boundary. Thus $\operatorname{meas}(R-R_h) \leqq \text{const. } h$. On the other hand, it is clear from (4) that

$$g_{hx} = -\int K_\xi(\xi - x, \eta - y)\{g(\xi, \eta) - g(x, y)\} dA_{\xi\eta} \leqq Mh^{-1}\omega(h),$$

where $\omega(t)$ is a modulus of continuity for $g(x, y)$. It follows easily from the pre-

ceding estimates that

$$I[g_h] - I[f_h] \leqq \text{const.}\ (h + \omega(h)),$$

and the proof is complete.

We conclude by showing that Lebesgue area is an additive function of sets (for continuously differentiable functions this is an obvious consequence of Th. 1).

THEOREM 3. *Let S_1 and S_2 be disjoint open regions with rectifiable boundaries. Let R_1 and R_2 be the closures of S_1 and S_2, and R the union of R_1 and R_2. Then*

$$L[f, R] = L[f, R_1] + L[f, R_2].$$

Proof. This theorem can be demonstrated directly from the definition of area, but an even more immediate proof can be given on the basis of Theorem 2. Indeed, by that result we have

$$L[f, R] - L[f, R_1] - L[f, R_2] = \lim_{h \to 0} \int_S \sqrt{(1 + f_{hx}^2 + f_{hy}^2)}\, dA,$$

where $S = R_h - R_{1h} - R_{2h}$. But, exactly as in the proof of Theorem 2, the last limit is zero.

Note. Since this paper is to some extent expository in character, we have not attempted to discuss refinements and extensions of the theory, nor have we paid particular attention to questions of attribution and priority. The reader interested in these matters may consult [3], [4], and [5], and the bibliographies listed there. The last named reference, in particular, includes an alternative definition of surface area which applies even to "surfaces" represented by locally integrable functions $f(x, y)$ defined over an arbitrary open region.

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SUCCESSOR AXIOMS FOR THE INTEGERS

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1. Introduction. There are two standard approaches to the integers. One is a set of axioms for an ordered integral domain whose set of positive elements is well-ordered. The other is the construction of the integers as equivalence classes of ordered pairs of natural numbers. In this paper the Peano axioms for the natural numbers are modified to provide an axiom set for the integers. In showing the adequacy of the axioms, we shall omit many details by assuming that the reader is familiar with the first few pages of Landau [1].

For the natural numbers based on the Peano axioms, the chief method of proof is induction. For the integers based on the modified Peano axioms, the chief method of proof is *symmetric induction*. In symmetric induction, the basis step may be given for any integer, and there are two induction steps: *forward*, from x to its successor x', and *backward*, from x' to x.

2. The axioms. The axioms for the set Z of integers are

1. Z is not empty.
2. To each integer x there is associated a unique integer x' (called the *successor* of x).
3. For all integers x and y, if $x'=y'$, then $x=y$.
4. For each integer y there is an integer x such that $x'=y$. (This x, which is unique by Axiom 3, is called the *predecessor* of y, and is denoted $'y$.)
5. If M is a set of integers such that (i) M is not empty; (ii) for every integer x, x is in M if and only if x' is in M; then $M=Z$.
6. There is a subset Q of Z such that (i) Q is not empty; (ii) for every integer x, if x is in Q, then x' is in Q; (iii) $Q\neq Z$.

For comparison and reference we now give the Peano axioms.

1*–3*. These are the same as 1–3, with "Z" replaced by "N" and "integer" replaced by "natural number".

4*. There is a natural number 1 such that for every natural number x, $x'\neq 1$.

5*. If M is a set of natural numbers such that (i) 1 is in M; (ii) for every natural number x, if x is in M, then x' is in M; then $M=N$.

Axiom 4 is the negation of 4*. From 3 and 4 follows

For every integer x, $'(x')=('x)'=x$.

Axiom 5 is symmetric induction. In the presence of the other axioms,

Axiom 5 *is equivalent to the statement obtained by replacing clause* (ii) *by*: *For every integer x, if x is in M, then x' and $'x$ are in M.* This equivalent of Axiom 5 we call the *second form of symmetric induction.*

Axiom 6 or some equivalent is necessary to rule out finite models, such as a model consisting of a, b and c, with $a'=b$, $b'=c$, $c'=a$. For the natural numbers, finite models are ruled out by 4* (in conjunction with 2* and 3*). Without Axiom 6 one cannot prove that $x\neq x'$, for a one-element model exists.

The axiom set for the integers differs from the Peano set in one respect, important from a logical standpoint. Axiom 6 cannot be translated into the symbolism of the first-order predicate calculus, in which quantification (for all x; there exists an x) is permitted only on elements. Axioms 5 and 5* are translated by a trick, in which the subset M is replaced by a property $P(x)$, and the universal quantifier "for all M" which is implicit, is omitted (as it must be), resulting in an infinite bundle of axioms, one for each $P(x)$. This trick does not work for Axiom 6 because the quantifier "there exists a Q" is an existential quantifier.

3. Addition and multiplication. In this section addition and multiplication are defined for Z, and Z is shown to be a ring. No use is made of Axiom 6. 0 is an arbitrary but fixed integer, and $1=0'$. To save repetition, we make the conventions that $a, b, \cdots, z$ always stand for integers, and every set is a set of integers.

Addition in Z is defined by the identities

$$x+0=x, \tag{1}$$

$$x+y'=(x+y)'. \tag{2}$$

A proof of existence and uniqueness of a binary composition satisfying (1) and (2) can be obtained by straightforward modification of the proof of Theorem 4 in Landau [1]. That the induction in [1] is based on 1 instead of 0 is inessential. The additional details for the backward induction steps can be supplied without difficulty.

Then, imitating Landau, the associative and commutative laws can be proved. There is no counterpart in [1] of the following theorem.

For every x there is a y such that $y+x=0$.

Proof. We use the second form of symmetric induction. Let M be the set of integers for which the theorem holds. 0 is in M since $0+0=0$. Assume x is in M and $y+x=0$. Then

$$'y+x'='('y+x)'=(x+'y)'=x+('y)'=x+y=0,$$

$$y'+'x='x+y'='(x+y)'=(y+'x)'=y+('x)'=y+x=0.$$

Hence $M=Z$.

Thus Z is an abelian group under addition. As usual, the negative of x is denoted $-x$, and $x-y$ is an abbreviation for $x+(-y)$. From (2) it follows that for every x, $x'=x+1$ and $'x=x-1$. Z is a cyclic group, because by Axiom 5, the integers

$$\cdots, -1-1, -1, 0, 1, 1+1, \cdots,$$

exhaust Z.

Multiplication is defined by the identities

$$x0 = 0, \tag{3}$$

$$x(y+1) = xy + x. \tag{4}$$

The proof of the theorem justifying addition is easily modified to prove the corresponding theorem for multiplication. Then Landau can be imitated to prove the commutative, distributive and associative laws.

As matters stand, Z is recognizable as the ring of integers modulo an unspecified n. To show $n=0$, Axiom 6 is necessary.

4. Order. In this section we show that Z is an ordered ring, and its set of positive elements satisfies the Peano axioms.

For the set Q of Axiom 6, there is an x such that x' is in Q and x is not in Q. Otherwise, $Q=Z$ by Axiom 5, contradicting Axiom 6. Let 0 be such an x and let $1=0'$. The 0 and 1 of Section 3 are now fixed to be this 0 and 1.

Let P be the intersection of all sets M such that
I. 1 is in M;
II. For every x, if x is in M, then x' is in M;
III. 0 is not in M.

P exists because the set Q of Axiom 6 is such an M. Then P has properties I–III, and is a subset of every set satisfying I–III.

P satisfies the Peano axioms, with the successor function of Z restricted to P as domain.

Proof. 1*–4* are easily verified. For 5*, let M be a subset of P satisfying the hypotheses of 5*. Then M satisfies I–III, making P a subset of M. Hence $M=P$.

We now show that P is a set of positive elements for Z. That P is closed under addition and multiplication is an easy exercise in ordinary (not symmetric) induction, which holds in P. In proving the trichotomy law, we shall use the fact that if x is in P, then $x=1$ or $x-1$ is in P. A proof of the trichotomy law follows and closes this paper.

For every integer x, exactly one of the following holds:
(1) $x=0$; (2) *x is in P*; (3) *$-x$ is in P.*

Proof. Neither the pair (1), (2) nor the pair (1), (3) can hold simultaneously because $0=-0$ is not in P. The pair (2), (3) is also impossible, because otherwise $x+(-x)=0$ is in P. Now let M be the set of all x for which at least one of (1)–(3) holds. We show $M=Z$ by symmetric induction. M is not empty because (1) holds for 0. Assume x is in M. If $x=0$, then $x+1=1$ is in P. If x is in P, then $x+1$ is in P. If $-x$ is in P, then either $-(x+1)=-x-1$ is in P, or $-x=1$, and

$x+1=0$. Assume $x+1$ is in M. If $x+1=0$, then $-x=1$ is in P. If $x+1$ is in P, then either x is in P, or $x=0$. If $-(x+1)$ is in P, then $-x$ is in P since $-x=-(x+1)+1$.

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PROPERTIES OF THE CANTOR SET AND SETS OF SIMILAR TYPE

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1. Introduction. Kestelman [2] proved the following theorem:

Let C be a closed bounded set in R_N (N-dimensional Euclidean space) having positive measure and let p be any positive integer. Then there exists a positive number δ (depending only on p and the measure of C) with the following property: if $\lambda_1, \cdots, \lambda_p$ are any vectors in R_N whose lengths are less than δ, then the set of all ξ such that $\xi \in C$, $\xi+\lambda_r \in C$ $(r=1, \cdots, p)$ is closed and has positive measure.

In the present note we propose to show that there exists a set E of zero measure with an almost similar property. We shall consider two special cases ((i) $N=2$, $p=1$; (ii) $N=N$, $p=1$) of the above theorem.

THEOREM 1. *Let C be the Cantor middle-third set in $[0, 1]$ and let E be the product $C \times C (= C^2)$ in R_2. (Obviously E is a nondense perfect set of Lebesgue plane measure zero.) Then if λ is a vector in R_2, whose length d is less than $\sqrt{2}$, the set of ξ such that $\xi \in E$, $\xi+\lambda \in E$, is nonempty for each of an infinite number (of power c) of directions of λ and the set of such ξ is closed for each particular direction of λ.*

Proof. Let λ be a vector such that $|\lambda| = d$, $0<d<\sqrt{2}$. We can find an infinite set of pairs (of power c), (d', d''), such that $d'^2+d''^2=d^2$, $0<d'<1$, $0<d''<1$. Since d' may take any value in some interval contained in $[0, 1]$, we can choose d' (the number of choices being infinite of power c) such that the set of pairs (x_1, x_2), where $x_1 \in C$, $x_2 \in C$, $x_2-x_1=d'$, has power c ([4], Ths. I, II). Having chosen d' in this way, we have now, corresponding to d'', at least one pair of points (y_1, y_2), $y_1 \in C$, $y_2 \in C$, $y_2-y_1=d''$ by Randolph's theorem [1].

Consider any pair (x_1, x_2) corresponding to the chosen d'. The points $F(x_1, y_1)$, $G(x_2, y_1)$, $H(x_2, y_2)$, $T(x_1, y_2)$ are points in E and form a rectangle in the unit square (Fig. 1). Now consider the vector $\lambda = \overrightarrow{FH}$, where $FH=d$ is a diagonal of the rectangle $FGHT$. Then $|\lambda| = d$ and λ makes an angle $\theta = \arctan(d''/d')$ with the x-axis. If F is the point ξ, then $\xi+\lambda$ is the point H, where $\xi \in E$, $\xi+\lambda \in E$. Since the set of pairs (x_1, x_2) has the power c, so does the set of rectangles like $FGHT$. It therefore follows that the vectors like $\overrightarrow{FH} (=\lambda)$ form an infinite set

of power c, each such vector being of length d, having the constant slope $\tan\theta$, and its endpoints in E.

Thus, since the number of choices for d' is infinite (of power c), with d fixed, we get an infinite number (of power c) of directions θ of λ. It should be noted that the vector $\overrightarrow{GT}$ also satisfies the conditions of the theorem, the direction in this case being $\pi-\theta$.

We now show that for each direction of λ for which the set of ξ such that $\xi\in E$, $\xi+\lambda\in E$, is nonempty, the set of all such ξ is closed. Let E_1 be obtained by translating E by an admissible vector $-\lambda$ (*cf.* [2], where $E_1\equiv T(E;\ -\lambda)$). Since each of E and E_1 is closed so is $E\cap E_1$, and this is precisely the set of ξ such that $\xi\in E$, $\xi+\lambda\in E$.

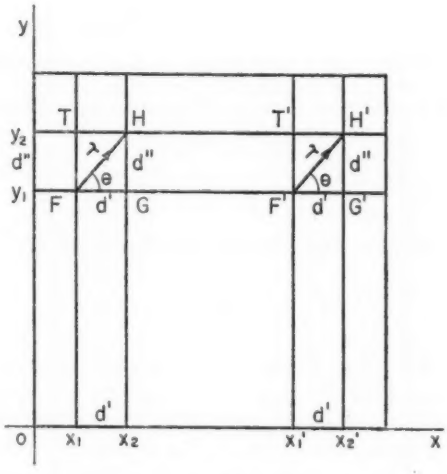

FIG. 1

Theorem 1 may easily be generalized as follows:

THEOREM 2. *Let C be the Cantor middle-third set in $[0, 1]$ and let E be the product $C\times\cdots\times C(=C^N)$ in R_N. Let $n\leqq N$ be any positive integer. Then if λ is a vector in R_N whose length is less than $\sqrt{n}$, the set of ξ, such that $\xi\in E$, $\xi+\lambda\in E$, is nonempty for each of an infinite number (of power c) of directions of λ, and the set of such ξ is closed for each particular direction of λ.*

Proof. After taking $|\lambda|=d$, we write $d_1^2+\cdots+d_n^2=d^2$, where at least one d_r has the same property as d' in Theorem 1 and $0<d_r<1$, $r=1, \cdots, n$. The proof is then completed exactly as above.

2. Consider the straight lines $y=3^{-n}x$, $n=0, 1, \cdots$, which cut the side KM of the unit square (Fig. 2) at $P(1, 3^{-n})$. Let $OP=\sqrt{\{(3^{2n}+1)/3^{2n}\}}$ be de-

noted by L and let θ be the angle that OP makes with the x-axis. Then we have the following:

THEOREM 3. *There is a subset of $E=C\times C(=C^2)$ lying on OP alone such that the distance set between points of this subset completely fills the closed interval* $[0, L]$.

Proof. Consider any l satisfying $0<l\leqq L$. Then $0<l\cos\theta\leqq L\cos\theta=1$. Therefore, by Randolph's theorem [1], we can find at least one pair of points (x_1, x_2), $x_1\in C$, $x_2\in C$ such that $l\cos\theta=x_2-x_1$. We draw ordinates through x_1 and x_2 to meet OP at $A(x_1, y_1)$ and $B(x_2, y_2)$, respectively. Now if

$$x = c_1 3^{-1} + c_2 3^{-2} + \cdots$$

is in C, then

$$y = 3^{-n}x = c_1 3^{-n-1} + c_2 3^{-n-2} + \cdots$$

is also in C. Therefore each of A and B lies on E and obviously $AB=l$.

There is, of course, a similar theorem for the straight lines $y=3^n x$ or $x=3^{-n}y$.

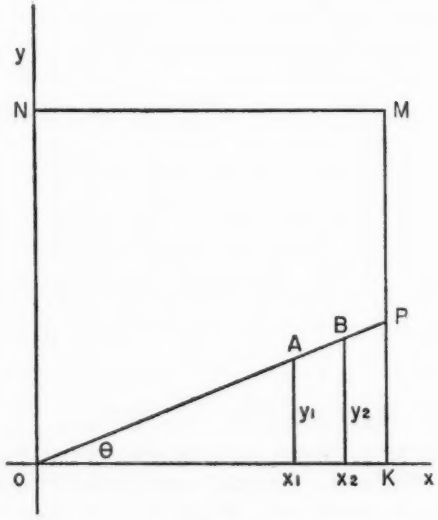

FIG. 2

3. In [3] two sets E_n' and E_n'', each of Cantor's type, were constructed on the x- and y-axes, respectively, in $0\leqq x\leqq 1$ and $0\leqq y\leqq 1$. The only difference was that the intervals were subdivided at each stage into $2n+1$ equal parts and the middle $(n+1)$th interval was suppressed. Each of E_n' and E_n'' is a nondense perfect set of measure zero. It was shown that for the product set $E_n=E_n'\times E_n''$, the distance set between its points completely fills the closed interval $[0, \sqrt{2}]$.

In this section we prove

THEOREM 4. *There are subsets of E_n, one lying on the diagonal OM and the other on the diagonal KN of the unit square* (Fig. 3) *such that the distance set between points of each subset completely fills the closed interval* $[0, \sqrt{2}]$.

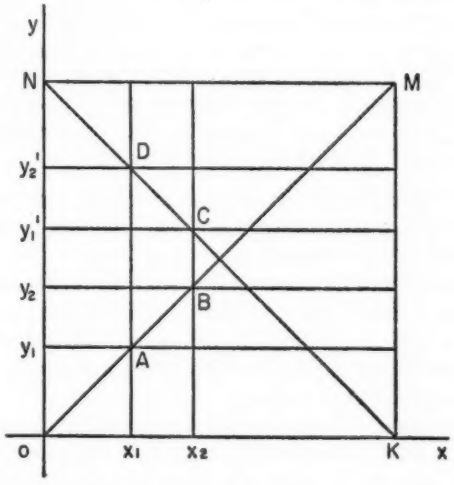

FIG. 3

Proof. Take the diagonal of the unit square joining $O(0, 0)$ and $M(1, 1)$ and consider any d satisfying $0 \leqq d \leqq \sqrt{2}$. Since $0 \leqq d/\sqrt{2} \leqq 1$ we can choose [3] at least one pair of points (x_1, x_2) on E_n' such that $x_2 - x_1 = d/\sqrt{2}$. Now $y_1 = x_1$ and $y_2 = x_2$ on E_n'' are such that $y_2 - y_1 = d/\sqrt{2}$. These points give rise to a square, two of whose corners A and B (obviously points on E_n) lie on the diagonal OM. It is evident that $AB = d$ so that the distance set between the points of E_n lying on OM completely fills $[0, \sqrt{2}]$.

It is, of course, easy to see that the distance set between the points of E_n lying on the other diagonal KN also fills $[0, \sqrt{2}]$.

I am grateful to Dr. H. M. Sen Gupta for his kind help and guidance in the preparation of this paper.

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ON THE DUAL OF A TRIVALENT MAP

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1. Introduction. In the literature on the four-color problem use is made of a "regular" map introduced by Heawood [1]. However, since "regular" now has quite another meaning, it has been suggested that maps with vertices of order three be called *trivalent*. The dual map of a trivalent map has the property that every country in such a map has three sides. Indeed, we may study the dual map as a triangular network. In this paper we shall study certain properties of the dual map of a trivalent map. This study leads very easily to certain known results, one of them a theorem due to Petersen [3].

2. Definitions and notation. In the literature on the four-color problem a term may be used with different meanings. To avoid misunderstanding, we shall explain the meaning of the terms we use.

(i) *Map.* Unless otherwise stated, all maps will be on a closed, simply connected surface and, without loss of generality, we may consider the simply connected surface as a sphere. A simple construction will transform any map on a sphere into a topologically equivalent map which may be drawn on a plane surface. The construction is as follows:

Suppose that the sphere is made up of a thin rubber sheet. Let the map over this surface contain $n+1$ countries C_k ($k=0, 1, \cdots, n$; $n>0$) with the boundary b_k. Cut out the country C_0 along the boundary b_0 and deform the map until all the countries C_k ($k\neq 0$) lie flat over a plane. This resulting "flat" map has n countries C_k' with the corresponding boundaries b_k'. Here C_k' and C_k and also b_k' and b_k are topologically equivalent. The country C_0' also exists in the flat map. The boundary contour of the flat map is precisely b_0', only one must imagine that C_0' now lies on the *reverse* side of the flat map. This construction of representing a spherical map on a plane surface is useful.

In general, a map M is divided into countries C_k by the corresponding boundaries b_k. These boundaries are closed curves. It may happen that b_s and b_t have some portion in common. Then C_s and C_t are called *adjacent*, otherwise they are *disjoint*. If b_s and b_t have only one point in common then C_s and C_t are also called disjoint. Since C_k is on a closed surface, the boundary b_k is divided into m mutually exclusive and exhaustive segments $b_{k_1}, \cdots, b_{k_m}$. When three or more sides meet, they can do so only at a point. This point is called a *vertex* of M. The *order* of a vertex V is the number of sides which meet at V.

(ii) *Trivalent map.* This is a map with each vertex of order three.

(iii) *Dual map.* Consider a map M with n countries C_k ($k=1, \cdots, n$) and with corresponding boundaries b_k. Take a point P_i in C_i. If C_i and C_j are adjacent then join P_i and P_j by an arc P_iP_j which cuts only once the common boundary of C_i and C_j but does not touch or intersect any other b_i in M. This construction will give a map M' with P_k ($k=1, \cdots, n$) as the vertices. M' is called a dual

map of M. It is easily seen that all dual maps of M are topologically equivalent. Hence we say that M' is the dual map (or simply dual) of M. We note that $(M')'$ is equivalent to M.

(iv) *Network.* Consider any map M with countries C_k having boundaries b_k. Let b_k be decomposed into b_{k_i} $(i=1, \cdots, m)$, where b_{k_i} is the ith side of C_k. Then the configuration $\{b_{k_i}\}$ for all permissible values of k and i, constitutes a *network.* This network is called *closed* if M is a map on a closed surface. Otherwise the network is *open.* A *path* in a network is a succession of sides in the network. The path is thus continuous and connects two or more vertices of M. A path is *closed* if the sides may be named cyclically so that only successive sides meet at a vertex. A closed path is called a *cycle.* Each b_k is a cycle. Cycles like the b_k are called *prime* cycles; we can not further decompose them into other cycles. Two cycles are *adjacent* when they have one or more common sides. A cycle is *even* when it has an even number of sides, otherwise it is *odd.* We may *add* or *unite* two or more adjacent cycles and get another cycle. In Figure 1 are shown

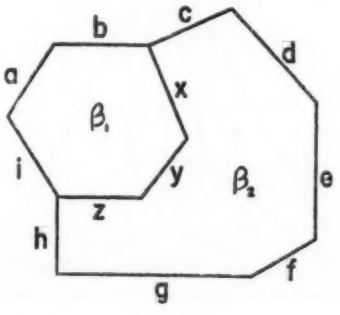

FIG. 1

two cycles β_1 and β_2. Here β_1 has sides a, b, x, y, z, i, while β_2 has sides $c, d, e, f, g, h, z, y, x$. We define $\beta_1+\beta_2$ as the cycle of sides $a, b, c, d, e, f, g, h, i$. By the product $\beta_1 \times \beta_2$ we denote the path of the sides x, y, z. The product is not a cycle unless β_1 and β_2 are the same cycles. If we denote an odd cycle by 1 and an even cycle by 0 then for adjacent cycles the following relations hold:

$$0+0=0, \quad 0+1=1, \quad 1+0=1, \quad 1+1=0.$$

In what follows, we shall mostly study *triangular* networks. A triangular network corresponds to a map which has all three-sided countries. The dual of a trivalent map contains all three-sided countries (see Lemma 1). Hence the network which corresponds to the dual of a trivalent map is triangular.

Every cycle in any network (whether closed or open) must be adjacent to at least one other cycle of the network; further, each side of a closed network belongs to at least two cycles.

3. LEMMA 1. *Let R' be the dual of a trivalent map R on a sphere. Then*

I. *All the countries in R' are three-sided.*

II. *The number of countries in R' is even.*

Proof. I. All the vertices of R are of order three. Consider a vertex V_1 in R. Let countries C_1, C_2, C_3, of R surround V_1. Thus, in R' there is a triangle, say T_1, with sides P_1P_2, P_2P_3, P_3P_1, and vertices P_1, P_2, P_3. Thus, to each vertex V_i there corresponds a triangle T_i.

II. For every closed network we have by Euler's formula $F-E+V=2$, where F, E, V are, respectively, the number of faces, sides and vertices of the network. But in the triangular network of R' we have $3F=2E$, since each triangular face has three sides and each side is common to two faces. Hence we get $F=2(V-2)$.

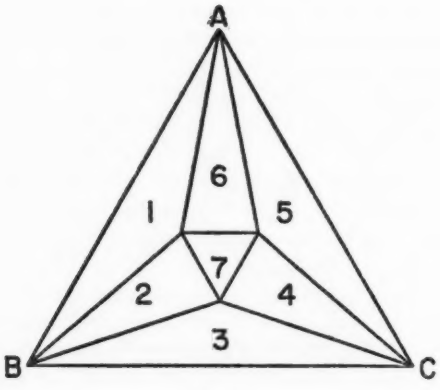

FIG. 2

The general appearance of the dual of a trivalent map is shown in Figure 2 for the dual of a trivalent map with six countries each of four sides. The number of triangles in the dual is $7+1=8$. The seven faces are numbered while the 8th face is the triangle on the *reverse* and is bounded by the cycle ABC. Figures 2 and 3 show the corresponding vertices of the closed trivalent map which may be taken as made up of the four-sided faces of a cube.

LEMMA 2. *Any open triangular network which contains an even number of triangles can be colored with two colors—red and green—in such a way that each triangle has one red side and two green sides while the boundary of the network has sides of only green color.*

Proof. Assume the result true for an open triangular network N_{2m} with $2m$ triangles. Consider an open triangular network N_{2m+2} with $2m+2$ triangles. Let β_{2m+2} be the boundary of N_{2m+2}. In N_{2m+2} select a triangle, say T_1, with boundary

b_1 so that β_{2m+2} and b_1 are adjacent. (Such a selection of T_1 is always possible). There must exist a triangle, say T_2, with boundary b_2 such that T_1 and T_2 are adjacent. Consider the cycle β_{2m} obtained from β_{2m+2} after deleting from it the sides $b_1 \times \beta_{2m+2}$, $b_2 \times \beta_{2m+2}$, $b_1 \times b_2$. Let β_{2m} be the boundary of the open triangular network N_{2m}. By the induction hypothesis, the network N_{2m} satisfies Lemma 2. Hence β_{2m} may be colored so that all sides are green. Color the sides in b_1+b_2 green and color $b_1 \times b_2$ red. Since $\beta_{2m+2}=(b_1+b_2)+\beta_{2m}$, the cycle β_{2m+2} gets all green sides and the network is colored as required. This establishes the lemma as the result is certainly true for $m=1, 2$.

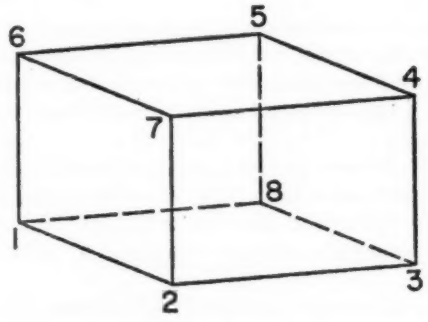

FIG. 3

THEOREM 1. *Any triangular network (whether closed or open) that contains an even number of triangles can be colored with two colors—red and green—so that each triangle has one red side and two green sides.*

Proof. The theorem is already proved above for an open triangular network with even number of triangles. Now consider a closed triangular network N_{2m+2} with $2m+2$ triangles. Take two adjacent triangles T_1 and T_2 with boundaries b_1 and b_2, respectively, in N_{2m+2}. Transform N_{2m+2} into a planar network by removing $T_1 \cup T_2$ from the surface and flattening out the remainder of the surface and its network N_{2m}. The boundary of N_{2m} is now b_1+b_2, and T_1, T_2, $b_1 \times b_2$ must be taken exterior to b_1+b_2. This *flat* network contains $2m$ triangles and is open. Hence, by Lemma 2, this network can be colored so that the boundary is all green and the triangles get one red side and two green sides each. But the boundary of N_{2m} is precisely the cycle b_1+b_2. Now color $b_1 \times b_2$ red and the theorem holds for the closed network N_{2m+2}.

COROLLARY 1. *Every closed triangular network can be colored with two colors—red and green—so that each triangle has one red side and two green sides.*

Proof. By Lemma 1, every closed triangular network has an even number of triangles. Hence by Theorem 1 we get the result.

COROLLARY 2 (PETERSEN'S THEOREM). *The network which corresponds to a trivalent map can be colored with two colors—red and green— so that at each vertex there meet one red side and two green sides.*

Proof. Let R be the trivalent map and R' its dual. At any vertex V in R there meet three sides, say s_1, s_2, s_3. Let s_j' in R' intersect s_j in R $(j=1, 2, 3)$. Color s_j and s_j' with the same color. Now a necessary and sufficient condition that Corollary 2 should hold is that Theorem 1 holds.

COROLLARY 3. *Every green cycle in a closed triangular network, colored according to Theorem* 1, *is even.*

Proof. The triangular network will be reduced to a quadrilateral network if we delete all the sides colored red. Then all prime cycles in this quadrilateral network are even since they have four sides each. Any cycle in this network is green and is obtained as the sum of two or more prime cycles. So, every cycle in the network is even.

The above corollary shows that it is possible to color any cycle of the quadrilateral network in two colors, say yellow and blue, so that the sides of the cycles are alternately yellow and blue. Unfortunately, this does not imply that *every* triangle of the triangular network has one red, one yellow, and one blue side.

4. LEMMA 3. *Four colors suffice to color a trivalent map R (and hence any map), so that no two adjacent countries of R have the same color, if the triangular network of R' can be colored with three colors so that each triangle has one side of each color.*

(This result is due to Tait [2]. He obtains it for the network of a trivalent map R. With suitable modification this result may be applied to the network of R'.)

Proof. Let V_i and V_j be two vertices of R'. There are cycles in R' which pass through V_i and V_j. There is at least one path between any two vertices of the network of R'. Consider the network of R' and let this be colored by three colors α, β, γ, so that each triangle of this network gets one side of each color. Now define a commutative group G of the four elements e, α, β, γ by the following multiplication table:

	e	α	β	γ
e	e	α	β	γ
α	α	e	γ	β
β	β	γ	e	α
γ	γ	β	α	e

Take four colors A, B, C, D to color the map R and suppose that these colors are obtained from one of them, say A, by the operations of the elements of G as

$$eA = A, \qquad \alpha A = B, \qquad \beta A = C, \qquad \gamma A = D.$$

The countries in R are represented by the vertices of R'. Hence the result will be established if we are able to assign colors to the vertices of R' so that no two vertices of the same triangle get like colors. We now proceed to assign the colors A, B, C, D to the vertices of R'. Let V_1, V_2, $\cdots$ be the vertices. Let V_1 receive color A. If the side V_1V_j has color θA ($\theta = \alpha, \beta, \gamma$) then place the color θA at V_j. Do so till all the vertices adjacent to V_1 are exhausted. Then take any colored vertex V_j ($j \neq 1$) and repeat the process, the only change to be noticed is that at V_j there is the color θA and not A. Repeat the process till all the vertices of R' are colored. It now remains to show that this scheme of assigning colors to the vertices does not lead to a contradiction, *i.e.*, no vertex gets two different colors. Consider two adjacent vertices say V_1 and V_2 which lie on a triangle T_0 having vertices V_1, V_2, V_3. Consider a path π between V_1 and V_2. Let π have a sides of the color αA, b sides of color βA and c sides of color γA. Then the *index* of π is the element θ given by

$$\theta = \alpha^a \beta^b \gamma^c.$$

We shall show that θ is invariant for all paths between V_1 and V_2. Let b_0 be the boundary of the triangle T_0. Each cycle through V_1 and V_2 can be obtained as the sum of prime-cycles (which are all triangles) one of them being b_0. Consider one such cycle β_m which is obtained by the sum of m prime cycles. Now β_m is the boundary of the open map M_m which is formed by the m triangles mentioned above. In M_m take a triangle T_1 with boundary b_1 such that β_m and b_1 are adjacent. We can decompose the cycle β_m into the two paths:

(i) the side V_1V_2,

(ii) the path which is made up of all the remaining sides in β_m.

Let the index of the first be ϕ (*i.e.*, ϕA is the color of the side V_1V_2). Let the index of the second path be ψ. From the map M_m delete T_1 to obtain the map M_{m-1} with boundary β_{m-1}. Clearly $\beta_m = \beta_{m-1} + b_1$. Let the sides in b_1 be x, y and z.

Now two cases arise:

Case 1: $\beta_m \times b_1$ has two sides, say y and z. Now β_m and β_{m-1} are different because of the sides of T_1. But the index of the path along the cycle β_{m-1} is equal to the index of the path along the cycle β_m because the contribution to the index due to any two sides of T_1 is equal to that due to the remaining third side of T_1.

Case 2: $\beta_m \times b_1$ has only one side, say z. A similar consideration is valid in this case.

Thus we conclude that the index of the path along β_{m-1} is the same as the index of the path along β_m. We repeat the process and reach the path along $\beta_1 = b_0$. Now the contribution due to sides V_1V_3 and V_3V_2 together is equal to

that due to the side V_1, and V_1, V_2 have the same index. What is true of V_1 and V_2 is true of any other pair of vertices of the network which proves the lemma.

If a triangular network is colored with three colors so that each side of a triangle has different color then we shall say that the network is *properly* colored. If a network has the property that it can be properly colored then we shall say that it is *properly colorable.*

COROLLARY 4. *Every cycle of a properly colored triangular network has index e.*

Proof. A cycle is a closed path from say V_i to V_i. Thus the index of a cycle must be such that it keeps the color at any of its vertices unaltered.

COROLLARY 5. *In a properly colored triangular network the cycles having sides of one color are even.*

Proof. The index of the cycle is e. Suppose all the sides of the cycle are of color α. Then $\alpha^m = e$ where m is the number of sides in the cycle. Hence m is even.

LEMMA 4. *If four colors are needed to color a trivalent map R then the triangular network of R′ is properly colorable.*

Proof. The above lemma is the converse of Lemma 3. Let the vertices V_i of R' be assigned colors A, B, C, D as they are found in R. Color the sides of the network of R' in the following scheme with colors α, β, γ:

	A	B	C	D
A	e	α	β	γ
B	α	e	γ	β
C	β	γ	e	α
D	γ	β	α	e

If V_i and V_j have colors X and Y then color the side V_iV_j with the color θ found at the intersection of X-row and Y-column in the above table. For example, if V_i is B and V_j is D then the side V_iV_j gets the color β. Since every side in the network of R' connects only one pair of vertices, it follows that each side of the network is assigned the color only once. It is now necessary to show that each triangle has one side of each color.

Suppose this is not true. Let a triangle T of vertices V_1, V_2, V_3 get two or more sides of the same color. Two cases are possible:

Case 1: V_1V_2, V_2V_3, V_3V_1 are all of the same color say α. Since V_1, V_2, V_3 are on the same triangle the colors at them are all different. If V_1 is A then

V_2 is $\alpha A = B$. But this is not possible. Hence our supposition that T has sides of the same color is wrong.

Case 2: Let V_1V_2 and V_1V_3 have the color α. In a similar way to that of Case 1, we may show that this is not possible.

Lemmas 3 and 4 apply to closed networks, but they also apply to an open network. For, given an open network, we first convert it into a closed network by introducing more triangles so that the "gaps" are filled in. Not until the open network has been converted into a closed network do we start the coloring. Then we may apply Lemmas 3 and 4 so that the closed network gets properly colored. Now, removing the additional triangles will not disturb the color scheme of the rest, which is certainly properly colored. Hence we get

THEOREM 2. *A necessary and sufficient condition that four colors suffice to color a trivalent map M (whether on a closed or an open surface) is that the triangular network of M' be properly colorable.*

In view of Theorem 2, it appears that a further study of the network of the dual of a trivalent map may be of interest.

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A GENERALIZED FIBONACCI SEQUENCE

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1. Introduction. Recently in this MONTHLY [1], [2], [3] there have appeared several problems and results involving the Fibonacci sequence, and consequently one is prompted to offer some comments on a generalized theory.

In the following development $a=\frac{1}{2}(1+\sqrt{5})$, $b=\frac{1}{2}(1-\sqrt{5})$ are the roots of $x^2-x-1=0$ so that $a+b=1$, $a-b=\sqrt{5}$, $ab=-1$. (The values of a and $-b$ are, of course, associated with the classical geometrical problem of the golden section.) For the Fibonacci sequence

(α) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, $\cdots$,

defined by the recurrence relation $F_n=F_{n-1}+F_{n-2}$ $(n\geqq 3)$ with $F_1=F_2=1$, it is well known (Daniel Bernoulli, 1732) that the nth term (Fibonacci number) is $F_n=(a^n-b^n)/\sqrt{5}$.

2. A generalized Fibonacci sequence. Suppose we preserve the recurrence relation but alter the first two terms to produce the generalized Fibonacci sequence defined by:

$$(\beta) \qquad H_n = H_{n-1} + H_{n-2} (n \geqq 3), \qquad H_1 = p, H_2 = p + q,$$

where p, q are arbitrary integers. That is, the generalized sequence is

$$(\gamma) \qquad p, p + q, 2p + q, 3p + 2q, 5p + 3q, 8p + 5q, 13p + 8q, \cdots.$$

Employing the usual method for difference equations, we deduce, after a little calculation, that

$$(\delta) \qquad H_n = \frac{1}{2\sqrt{5}} (la^n - mb^n),$$

where $l=2(p-qb)$, $m=2(p-qa)$, so that $l+m=2(2p-q)$, $l-m=2q\sqrt{5}$, $\frac{1}{4}lm = p^2-pq-q^2=e$ (say).

Certain results follow almost immediately from (β) and/or (δ), *viz.*,

$$(1) \qquad H_n/H_{n-1} \to a, \qquad H_n/H_{n-i} \to a^i, \qquad H_n/F_n \to p - qb \quad (\text{as } n \to \infty);$$

$$(2) \qquad 2^{n+1}H_n = \frac{l - m}{\sqrt{5}} \sum_{i=0}^{[\frac{1}{2}n]} 5^i C_{2i}^n + (l + m) \sum_{i=0}^{[\frac{1}{2}n-\frac{1}{2}]} 5^i C_{2i+1}^n;$$

$$(3) \qquad H_{n+2} - 2H_n - H_{n-1} = 0, \qquad H_{n+1} - 2H_n + H_{n-2} = 0;$$

$$(4) \qquad \sum_{i=0}^{n-1} H_{2i+1} = H_{2n} - q, \qquad \sum_{i=1}^{n} H_{2i} = H_{2n+1} - p;$$

$$(5) \qquad \sum_{i=1}^{n} (H_{2i-1} - H_{2i}) = - H_{2n-1} + p - q.$$

Writing $\sigma_n = \sum_{i=1}^{n} H_i$, $\tau_n = \sum_{i=1}^{n} \sigma_i$, we have

$$(6) \qquad \sigma_n = H_{n+2} - H_2, \qquad \tau_n = H_{n+4} - (n + 2)H_2 - H_1.$$

From (γ) we observe that

$$(7) \qquad H_{n+1} = qF_n + pF_{n+1}, \qquad H_{n+2} = pF_n + (p + q)F_{n+1}.$$

Putting $n=r$ in (7) and using (β), we find in turn

$$H_{r+3} = (p + q)F_r + (2p + q)F_{r+1} = H_2F_r + H_3F_{r+1},$$
$$H_{r+4} = (2p + q)F_r + (3p + 2q)F_{r+1} = H_3F_r + H_4F_{r+1}, \cdots,$$

and, in general,

$$(8) \qquad H_{n+r} = H_{n-1}F_r + H_nF_{r+1} \qquad (n \geqq 3).$$

In the ensuing results it is perhaps preferable to use (δ) throughout, in which

case the following identities may be advantageously used:

$$a - \frac{1}{a^2} = -b\sqrt{5}, \qquad b - \frac{1}{b^2} = a\sqrt{5};$$

$$(\epsilon) \qquad a + \frac{1}{a} = \sqrt{5}, \qquad b + \frac{1}{b} = -\sqrt{5};$$

$$a^2 - a - 1 = 0, \qquad b^2 - b - 1 = 0.$$

(No attempt at any meaningful order in these results is implied.)

$$(9) \qquad H_{n-1}^2 + H_n^2 = (2p - q)H_{2n-1} - eF_{2n-1},$$

$$(10) \qquad H_{n+1}^2 - H_{n-1}^2 = (2p - q)H_{2n} - eF_{2n},$$

$$(11) \qquad H_{n-1}H_{n+1} - H_n^2 = (-1)^n e,$$

$$(12) \qquad H_nH_{n+r+1} - H_{n-s}H_{n+r+s+1} = (-1)^{n+s}eF_sF_{r+s+1},$$

$$(13) \qquad H_n^3 + H_{n+1}^3 = 2H_nH_{n+1}^2 + (-1)^n e,$$

$$(14) \qquad H_{n+1-r}H_{n+1+r} - H_{n+1}^2 = (-1)^{n-r}eF_r^2.$$

(Observe that (11) is a special case of (14) when $r=1$ and n is replaced by $n-1$.) Putting $r=n$ in (14), we derive

$$(15) \qquad H_{n+1}^2 + eF_n^2 = pH_{2n+1}.$$

Numerous other results may be deduced but, as we are not concerned here with an exhaustive list, we mention only the useful "Pythagorean" theorem:

$$(16) \qquad \{2H_{n+1}H_{n+2}\}^2 + \{H_nH_{n+3}\}^2 = \{2H_{n+1}H_{n+2} + H_n^2\}^2,$$

and the remarkable fact that

$$(17) \qquad \frac{H_{n+r} + (-1)^rH_{n-r}}{H_n} = F_{r+1} + (-1)^rF_{r-1} = \left(\frac{a}{2}\right)^r + (-1)^r\left(\frac{2}{a}\right)^r,$$

i.e., the expression on the left-hand side of (17) is independent of p, q, and n.

Taking $p=3$, $q=1$, for instance, so that (γ) becomes

$$(\gamma) \qquad 3, 4, 7, 11, 18, 29, 47, 76, 123, \cdots$$

and putting $n=3$, we find that (16) yields $396^2+203^2=445^2$. Again using these values for p and q, and setting $n=7$, $r=2$ in (17), we have $(123+18)/47=3=F_3+F_1$. Note that the simple results $3^2+4^2=5^2$, $5^2+12^2=13^2$, occur when $p=1$, $q=0$, $n=1$, and $p=1$, $q=0$, $n=2$, respectively. When $p=1$, $q=0$, $n=3$, we obtain $8^2+15^2=17^2$ after simplifying and dividing throughout by 4. An interesting question is: Does (16) exhaust all the Pythagorean number triples?

Searching through the available literature on generalizations of (α), one sees that, broadly speaking, the work may be generalized in two main directions. Either the recurrence relation can be generalized and extended (and this has been done in a variety of ways) or the recurrence relation is preserved but the first two Fibonacci numbers are altered—this has been our approach. (Naturally, these two techniques could be combined.) The research to which this article most closely approximates seems to be that of Tagiuri ([5], p. 404), who wrote in Periodico di Matematiche, vol. 16, 1901, on the subject. The results (7), (12), (14), (17) above are due to Tagiuri, who used a slightly different notation from that used here. Doubtless, a good deal of work along these lines has been done by mathematicians not recorded in [5].

3. Special cases. The Fibonacci sequence (α) is obtained from (γ) by putting $p=1$, $q=0$. Making this simplification, so that $l=m=2$, $e=1$, and (δ) reduces to $F_n=(a^n-b^n)/\sqrt{5}$, we obtain, for instance, from (9), (11), and (15), respectively, the well-known results

$$F_{n-1}^2+F_n^2=F_{2n-1}, \tag{9'}$$

$$F_{n-1}F_{n+1}-F_n^2=(-1)^n, \tag{11'}$$

$$F_{n+1}^2+F_n^2=F_{2n+1}, \tag{15'}$$

attributed to Catalan, Simson, and Lucas, respectively. Obviously, (15′) is obtainable from (9′) by replacing n by $n+1$, but we cannot say the same about (15) and (9).

All the commonly known properties of the Fibonacci sequence are thus deducible from the generalized sequence as special cases when $p=1$, $q=0$. For example, if p is arbitrary and $q=1$ in (γ), we obtain the variant of the Fibonacci sequence discussed by Guest [4]. With his further modification $p=10$, we find that his $F_{11}=89=e$ in our notation and all the results given by him then occur as special cases of the general theory when $p=10$, $q=1$.

A question that may well be asked is: Under what conditions (relating to p and q) is the Fibonacci sequence (α) repeated? Obviously, from (γ) for $H_1=H_2=p$ (*i.e.*, $q=0$), each term of (α) is merely multiplied by p as Guest observed. Furthermore, we must avoid the case when q and p are consecutive Fibonacci numbers for if $p=F_n$, $q=F_{n-1}$, we find that the new sequence is the Fibonacci sequence with the first $n-1$ terms missing, *i.e.*,

$$F_n,\ F_n+F_{n-1},\ 2F_n+F_{n-1},\ 3F_n+2F_{n-1},\ 5F_n+3F_{n-1},\ \cdots,$$

which is the same as F_n, F_{n+1}, F_{n+2}, F_{n+3}, F_{n+4}, $\cdots$. In particular, if $p=8$, $q=5$, the new sequence is 8, 13, 21, 34, 55, *i.e.*, (α) with the first five numbers missing.

On the other hand, if $p=F_{n-1}$, $q=F_n$, we do not obtain the Fibonacci sequence. With $p=5$, $q=8$, for example, we get the sequence 5, 13, 18, 31, 49, 80, 129, $\cdots$.

When $q=np$ (n an integer), we have the sequence

$$p\{1, 1+n, 2+n, 3+2n, 5+3n, 8+5n, 13+8n, \cdots\},$$

i.e., p times the generalized sequence with $p=1$, $q=n$.

Turning now to some recent problems and remarks on the ordinary Fibonacci sequence, we can generalize Ivanoff's result [2] thus:

$$\sum_{i=0}^{n} C_i^n H_{n-i} = \frac{1}{2\sqrt{5}}\left\{ l \sum_{i=0}^{n} C_i^n a^{n-i} - m \sum_{i=0}^{n} C_i^n b^{n-i} \right\}$$

$$= \frac{1}{2\sqrt{5}} \{ l(1+a)^n - m(1+b)^n \}$$

$$= \frac{1}{2\sqrt{5}} \{ la^{2n} - mb^{2n} \} = H_{2n},$$

where in the next to the last step we have used (ϵ).

Danese's result [3] is now seen to be a special case of (12) when n is replaced by $n+h$ and $s=h$, $r=k-h-1$ (with $p=1$, $q=0$, of course). The last two results in Ganis [1] are particular cases of Danese's result, as noted in [3]. Venkannayah's problem [3] carries over directly to the generalized case, but Everman's problem [3] is a special case (remembering that $F_5=5$) of a theorem due to Tagiuri, *viz.*, that F_r is a multiple of F_s provided that r is a multiple of s, which is not valid in the generalized sequence. For example, $H_6/H_3=29/7$.

Of the remaining results quoted in [1] (and due originally to Simson, Lucas, and Piccioli), the first and third are generalized in (11) and (6), respectively, and the second, $F_{n+1}=C_0^n+C_1^{n-1}+C_2^{n-2}+\cdots$, could be considered to be generalized in (7).

This paper developed out of an interest in the Fibonacci sequence and a desire to extend the results of Guest's stimulating article. Ever since Fibonacci (Leonardo of Pisa) wrote his Liber Abbaci in 1202, his intriguing sequence has fascinated men through the centuries, not only for its inherent mathematical riches, but also for its applications in art and nature. Indeed, it is almost true to say that the research generated by its nearly amounts to the quantity of offspring generated by the mythical pair of rabbits who started Fibonacci off on the problem!

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MATHEMATICAL NOTES

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MOMENTS OF A FUNCTION ON THE CANTOR SET

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If a set of points x comprises a countable set x_i, then the average value of a function $f(x)$ on these points would be defined as

$$u_0(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i).$$

If the points x comprise a set S of positive measure m, then $u_0(f)$ would be defined as $u_0(f) = (1/m)\int_S f(x)dx$. For the Cantor set neither of these alternatives holds, so a new definition is needed. The appended reference considers a very similar problem, but we will use a different approach.

Observe that as the variable x ranges over points of the Cantor set (on the unit interval), the variables $\frac{1}{3}x$ and $\frac{2}{3}+\frac{1}{3}x$ range over those points of the Cantor set on the first third and last third of the unit interval, respectively. This suggests the relation

(1) $\quad u_0(f) = \frac{1}{2}u_0(f_1) + \frac{1}{2}u_0(f_2)$, where $f_1 = f(\frac{1}{3}x)$ and $f_2 = f(\frac{2}{3} + \frac{1}{3}x)$.

Now we can define $u_0(f)$ as a linear functional, with $u_0(1) = 1$, which satisfies relation (1). As a linear functional, if $g(x) = a_0 + a_1x \cdots + a_nx^n$, then $u_0(g) = a_0u_0(1) + a_1u_0(x) + \cdots + a_nu_0(x^n)$, which is consonant with the notion of average or moment. By means of this definition, we obtain values of $u_0(x^n)$, $n = 1, 2, \cdots$, as solutions of successive equations.

These results may be used in turn, to obtain average values of polynomial functions and also "higher" moments.

Examples.

$$u_0(x) = \bar{x} = \frac{1}{2}\cdot\frac{1}{3}\,\bar{x} + \frac{1}{2}\left(\frac{2}{3} + \frac{1}{3}\,\bar{x}\right), \qquad \bar{x} = \frac{1}{2}\cdot$$

$$u_0(x^2) = \overline{x^2} = \frac{1}{2}\,u_0\left\{\left(\frac{1}{3}\,x\right)^2\right\} + \frac{1}{2}\,u_0\left\{\left(\frac{2}{3} + \frac{1}{3}\,x\right)^2\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{9}\,u_0(x^2) + \frac{4}{9}\,u_0(1) + \frac{4}{9}\,u_0(x) + \frac{1}{9}\,u_0(x^2)\right\}$$

$$= \frac{1}{2}\left\{\frac{2}{9}\,\overline{x^2} + \frac{4}{9} + \frac{2}{9}\right\}, \qquad \text{since } u_0(x) = \bar{x} = \frac{1}{2};$$

whence $\overline{x^2} = 3/8$.

$$u_0(x^3) = \overline{x^3} = \frac{1}{2}\cdot\frac{1}{27}\overline{x^3} + \frac{1}{2}\left(\frac{8}{27} + \frac{4}{9}\cdot\frac{1}{2} + \frac{2}{9}\cdot\frac{3}{8} + \frac{1}{27}\overline{x^3}\right), \qquad \overline{x^3} = \frac{5}{16}\cdot$$

$$u_0(x^4) = \overline{x^4} = \frac{1}{2}\cdot\frac{1}{81}\overline{x^4} + \frac{1}{2}\left(\frac{16}{81} + \frac{32}{81}\cdot\frac{1}{2} + \frac{8}{27}\cdot\frac{3}{8} + \frac{8}{81}\cdot\frac{5}{16} + \frac{1}{81}\overline{x^4}\right);$$

whence $\overline{x^4} = 87/320, \cdots$.

$$u_0(e^{-x}) = \overline{e^{-x}} = 1 - \frac{1}{2} + \frac{3}{8}\cdot\frac{1}{2!} - \frac{5}{16}\cdot\frac{1}{3!} + \frac{87}{320}\cdot\frac{1}{4!} - + \cdots = .646, \cdots.$$

The average values can be used to obtain higher-order moments:

$$u_1(x) = \overline{x^2}/\bar{x} = 3/4 \quad u_2(x) = \overline{x^3}/\bar{x} = 5/8 \quad u_1(x^2) = \overline{x^3}/\overline{x^2} = 5/6, \cdots.$$

Note: $u_0(x^n) = m_n$ of the reference.

Reference

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A NOMOGRAPHIC SOLUTION OF THE QUARTIC

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Since the general quartic equation

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \tag{1}$$

contains five homogeneous or four essential coefficients in addition to the unknown x, it is clear that no simple nomogram can be constructed for its solution. However, if λ is suitably chosen, the substitution $x = \lambda t$ will reduce (1) to a form in which only three arbitrary coefficients appear, and an interesting nomographic solution for the transformed equation becomes possible. Specifically, if we set

$$x = \begin{cases} \sqrt{(a_3/a_1)}\,t & a_1a_3 > 0, \\ \sqrt{(-a_3/a_1)}\,t & a_1a_3 < 0, \end{cases} \tag{2}$$

we obtain

$$Ut^4 + t^3 + Vt^2 \pm t + W = 0, \tag{3}$$

where

$$U = \frac{a_0}{a_1}\sqrt{\left(\pm\frac{a_3}{a_1}\right)}, \qquad V = \frac{a_2}{a_1}\sqrt{\left(\pm\frac{a_1}{a_3}\right)}, \qquad W = \pm\frac{a_4}{a_3}\sqrt{\left(\pm\frac{a_1}{a_3}\right)},$$

the plus sign applying if $a_1a_3>0$, the minus sign if $a_1a_3<0$.* Obviously, the coefficients in (3) can be obtained from those in (1) by easy slide rule calculations. We now write (3) in the form $Ut^4+(t^3\pm t)=-Vt^2-W=Z$ and consider the possibility of constructing nomograms for the component equations

$$Ut^4 + (t^3 \pm t) - Z = 0, \tag{4.1}$$

$$Vt^2 + Z + W = 0, \tag{4.2}$$

which can be used simultaneously to solve (3), and hence (1).

Equation (4.1) contains just three variables and the construction of a nomogram for its solution presents no problem. Equation (4.2) contains four variables, but if we treat $(Z+W)$ as a single quantity this equation, too, can be solved nomographically. The necessary nomograms can be constructed in infinitely many ways, but since it turns out to be desirable to have the scales of Z and $(Z+W)$ linear and congruent, our choice is essentially limited to the nomograms derived from the following determinantal equivalents of (4.1) and (4.2):

$$\begin{vmatrix} \dfrac{t^3 \pm t}{t^4+1} & \dfrac{t^4}{t^4+1} & 1 \\ -U & 1 & 1 \\ Z & 0 & 1 \end{vmatrix} = 0, \tag{5.1}$$

$$\begin{vmatrix} 0 & \dfrac{t^2}{t^2+1} & 1 \\ V & 1 & 1 \\ Z+W & 0 & 1 \end{vmatrix} = 0. \tag{5.2}$$

These nomograms are shown in Figure 1† and are to be used in the following way. First, the nomograms are placed so that the Z- and $(Z+W)$-scales fall along the same line and the origin of the Z-scale falls at the value W on the $(Z+W)$-scale. Then lines passing respectively through the appropriate points on the U- and V-scales and intersecting on the common $(Z, Z+W)$-scale are determined so that they intersect the respective t-scales (choosing the proper t-scale in the upper nomogram according as $a_1a_3>0$ or $a_1a_3<0$) in the same value of t. Since the Z- and $(Z+W)$-scales are uniformly and congruently graduated, it is obvious that values of t determined as we have just described will be solutions of (3) and will lead, through (2), to solutions of (1).

* If $a_1a_3=0$, (1) initially contains only three essential coefficients and can therefore be handled directly by an obvious modification of the method we shall use on (3).

† Because of the near coincidence of the two t-curves around $t=0$ and $t=\infty$ in the upper nomogram, it would be more practical, though less elegant, to display these scales in separate nomograms.

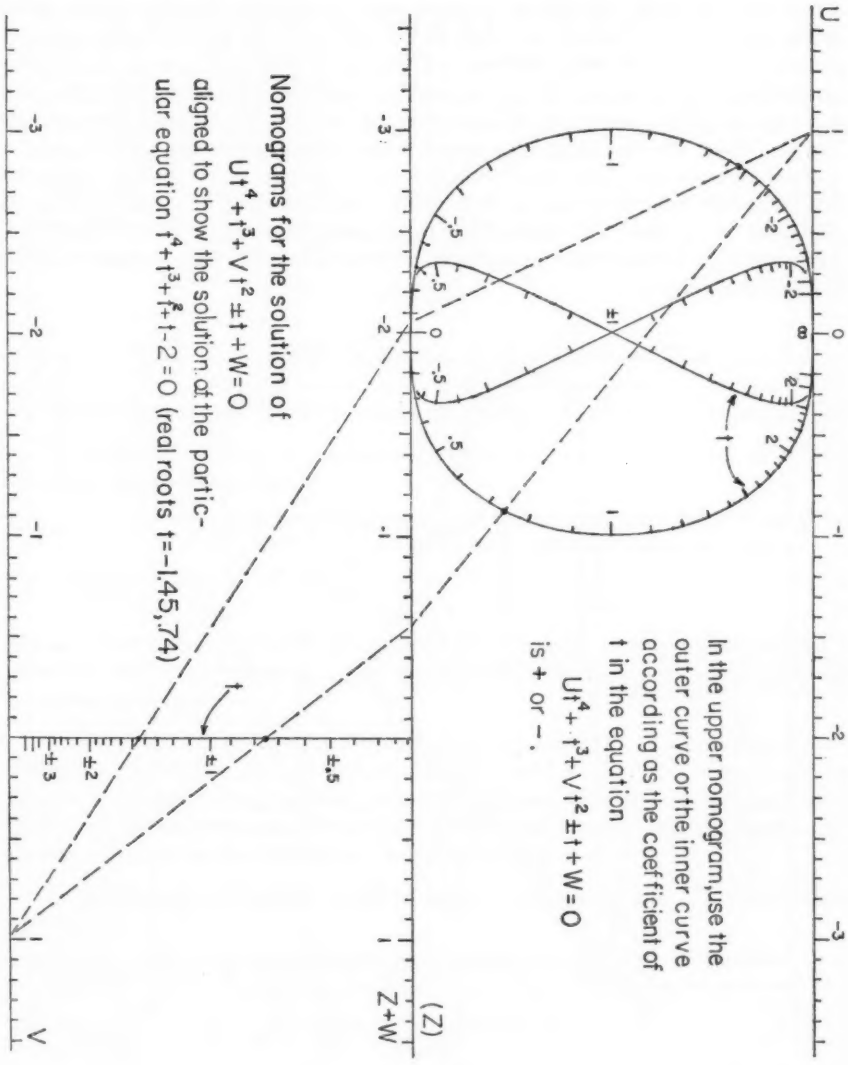
Nomograms for the solution of
$Ut^4+t^3+Vt^2\pm t+W=0$
aligned to show the solution at the particular equation $t^4+t^3+t^2+t-2=0$ (real roots t=-1.45,.74)
In the upper nomogram, use the outer curve or the inner curve according as the coefficient of t in the equation
$Ut^4+t^3+Vt^2\pm t+W=0$
is + or −.
U
V
Z+W
(Z)

FIG. 1

This process of trial and error can be systematized by means of the apparatus illustrated in Figure 2. This consists of a shallow bed, *LMNO*, bearing sliding panels, *ABCD* and *EFGH*, guided as shown, to which prints of the nomograms are attached so that the linear scales coincide with the edges of the panels.

Clips bearing small pins placed so that their centers fall directly on the outer scales can be moved along the edges of the bed until the pins coincide with the points P_U and P_V. Between the panels a triangular bar bearing a pin at one end slides back and forth along the common Z- and $(Z+W)$-scales. Finally, two transparent arms, slotted to fit over the pins at P_U and P_V are hinged at the pin P_Z. When the clips have been properly located and the upper panel has been shifted through the distance W relative to the lower, it is a simple matter to draw the center bar back and forth until hair lines inscribed on the two arms are observed to cut the two t-scales in points associated with the same value of t, and a solution is found. A working model of this device has been constructed and its utility confirmed.

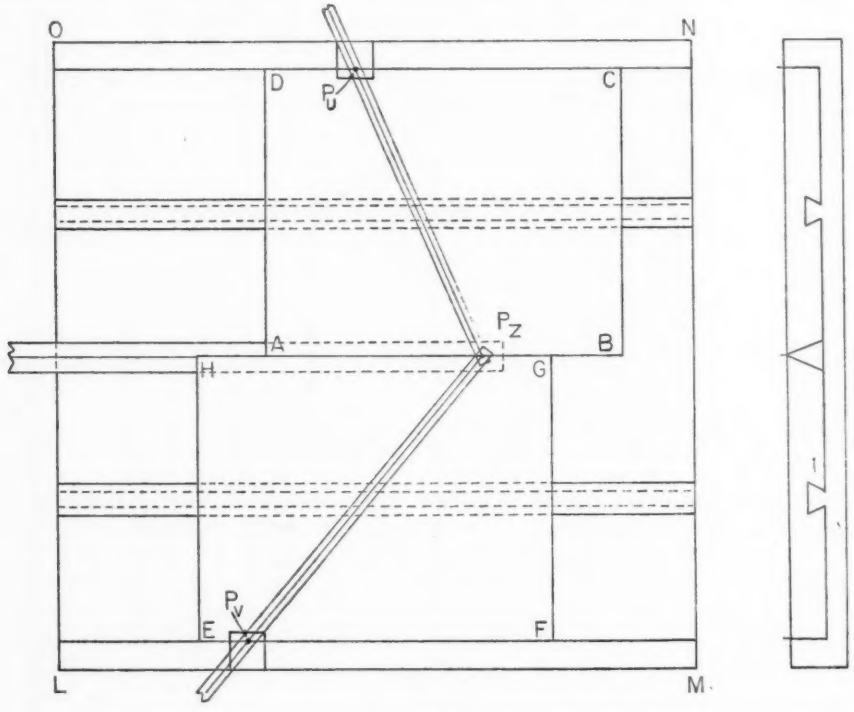

FIG. 2

In conclusion, it is worth pointing out that the procedure discussed in this note, in conjunction with the mechanical device we have just described, can also be used to solve the more general four-variable equation, $UT_1(t)+VT_2(t)+T_3(t)+W=0$, of which (3) is just a special case.

AN EXISTENCE THEOREM FOR INFINITE MATRICES

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1. Our principal object in the present note is to find necessary and sufficient conditions for the existence of infinite matrices subject to certain linear constraints. We shall, in fact, establish the following result.

THEOREM 1. *Let d_k, ρ_k, σ_k $(k \geqq 1)$ be real numbers. Then there exists an infinite matrix with nonnegative elements, with diagonal elements $d_1, d_2, \cdots$, with row-sums $\rho_1, \rho_2, \cdots$, and with column-sums $\sigma_1, \sigma_2, \cdots$ if and only if*

$$0 \leqq d_k \leqq \min(\rho_k, \sigma_k) \qquad (k \geqq 1), \tag{1}$$

$$\sup_{j \geqq 1} (\rho_j + \sigma_j - 2d_j) \leqq \sum_{k=1}^{\infty} (\rho_k - d_k) = \sum_{k=1}^{\infty} (\sigma_k - d_k) \leqq \infty.$$

A more general result which can be deduced from Theorem 1 is as follows.

THEOREM 2. *Let m be a nonnegative integer, and let d_k, ρ_k, σ_k $(k \geqq 1)$ be real numbers. Then the conditions*

$$0 \leqq d_k \leqq \min(\rho_k, \sigma_{m+k}) \qquad (k \geqq 1),$$

$$\sup_{j \geqq 1} (\rho_j + \sigma_{m+j} - 2d_j) \leqq \sum_{k=1}^{\infty} (\rho_k - d_k) = \sum_{k=1}^{m} \sigma_k + \sum_{k=1}^{\infty} (\sigma_{m+k} - d_k) \leqq \infty \tag{2}$$

are necessary and sufficient for the existence of an infinite matrix with nonnegative elements, with row-sums $\rho_1, \rho_2, \cdots$, with column-sums $\sigma_1, \sigma_2, \cdots$, and with mth superdiagonal $d_1, d_2, \cdots$.

In Sections 2 and 3 we shall give a proof of Theorem 1. The derivation of Theorem 2 from Theorem 1 is straightforward, and we shall therefore omit the details of the argument. In Section 4 the special case of infinite doubly stochastic matrices will be examined. Finally, a further specialization will lead to the discussion of *finite* doubly stochastic matrices in Section 5.

2. The proof of Theorem 1 will be seen to depend on the following simple lemma.

LEMMA. *Let ρ_k, σ_k $(k \geqq 1)$ be nonnegative numbers such that*

$$\sum_{k=1}^{\infty} \rho_k = \sum_{k=1}^{\infty} \sigma_k = s < \infty, \tag{3}$$

$$\rho_k + \sigma_k \leqq s \qquad (k \geqq 1). \tag{4}$$

Then there exist numbers y_k, z_k $(k \geqq 2)$ such that

$$\sum_{k=2}^{\infty} y_k = \rho_1, \tag{5}$$

$$\sum_{k=2}^{\infty} z_k = \sigma_1, \tag{6}$$

$$0 \leqq y_k \leqq \sigma_k \qquad (k \geqq 2), \tag{7}$$

$$0 \leqq z_k \leqq \rho_k \qquad (k \geqq 2), \tag{8}$$

$$y_k + z_k \geqq \rho_k + \sigma_k + \rho_1 + \sigma_1 - s \qquad (k \geqq 2). \tag{9}$$

We write $\tau = \sum_{k=2}^{\infty} \max(0, \rho_k+\sigma_k+\rho_1-s)$, so that $\tau = \sum_{k=2}^{\infty} \delta_k(\rho_k+\sigma_k+\rho_1-s)$, where each δ_k is 1 or 0. We shall show that

$$\tau \leqq \rho_1. \tag{10}$$

When all δ's are 0, this is obvious; when just one δ is positive, this follows by (4). When the number of positive δ's is at least two, we have

$$\begin{aligned}\tau &= \sum_{k=2}^{\infty} \delta_k(\rho_k + \sigma_k) + (\rho_1 - s) \sum_{k=2}^{\infty} \delta_k \\ &\leqq \sum_{k=2}^{\infty} (\rho_k + \sigma_k) + 2(\rho_1 - s) \\ &= s - \rho_1 + s - \sigma_1 + 2(\rho_1 - s) = \rho_1 - \sigma_1 \leqq \rho_1.\end{aligned}$$

Thus (10) is proved. Hence, in view of (4), there exist numbers y_k $(k \geqq 2)$ satisfying (5) and

$$\max(0, \rho_k + \sigma_k + \rho_1 - s) \leqq y_k \leqq \sigma_k \qquad (k \geqq 2). \tag{11}$$

Now put $\tau' = \sum_{k=2}^{\infty} \max(0, \rho_k+\sigma_k+\rho_1+\sigma_1-s-y_k)$, so that

$$\tau' = \sum_{k=2}^{\infty} \epsilon_k(\rho_k + \sigma_k + \rho_1 + \sigma_1 - s - y_k),$$

where each ϵ_k is 1 or 0. If the number of positive ϵ's is at least two, then

$$\begin{aligned}\tau' &= \sum_{k=2}^{\infty} \epsilon_k(\rho_k + \sigma_k - y_k) + (\rho_1 + \sigma_1 - s) \sum_{k=2}^{\infty} \epsilon_k \\ &\leqq \sum_{k=2}^{\infty} (\rho_k + \sigma_k - y_k) + 2(\rho_1 + \sigma_1 - s) \\ &= s - \rho_1 + s - \sigma_1 - \rho_1 + 2(\rho_1 + \sigma_1 - s) = \sigma_1.\end{aligned}$$

Thus we have

$$\sum_{k=2}^{\infty} \max(0, \rho_k + \sigma_k + \rho_1 + \sigma_1 - s - y_k) \leqq \sigma_1; \tag{12}$$

and, in view of (11), this inequality is still valid when the number of positive ϵ's is less than two.

Finally, by (12) and (4), there exist numbers z_k $(k \geqq 2)$ which satisfy (6), (8), and (9). This establishes the lemma.

3. We now come to the proof of Theorem 1. Suppose, in the first place, that there exists a matrix of the type described in the theorem. Then (1) is obviously satisfied. Moreover, for any $j \geqq 1$, $\rho_j + \sigma_j - 2d_j$ is the sum of the nondiagonal elements in the jth row and jth column, while

$$\sum_{k=1}^{\infty} (\rho_k - d_k) = \sum_{k=1}^{\infty} (\sigma_k - d_k) \quad (\leqq \infty)$$

is the sum of *all* nondiagonal elements. Hence (2) follows.

Next, suppose that (1) and (2) are given. We have to show that there exists a matrix with the requisite properties. It may clearly be assumed without loss of generality that $d_k = 0$ $(k \geqq 1)$. Thus ρ_k, σ_k $(k \geqq 1)$ are given nonnegative numbers such that

$$\rho_j + \sigma_j \leqq \sum_{k=1}^{\infty} \rho_k = \sum_{k=2}^{\infty} \sigma_k \leqq \infty \qquad (j = 1, 2, \cdots). \tag{13}$$

We denote by s the common value of the sums of the infinite series in (13). We have now to show that there exists an infinite matrix with nonnegative elements, with zero diagonal elements, with row-sums $\rho_1, \rho_2, \cdots$, and with column-sums $\sigma_1, \sigma_2, \cdots$.

Case 1. $s = \infty$.

It is clearly possible to choose numbers x_{1k}, x_{k1} $(k \geqq 2)$ such that

$$\sum_{k=2}^{\infty} x_{1k} = \rho_1, \qquad \sum_{k=2}^{\infty} x_{k1} = \sigma_1, \tag{14}$$

$$0 \leqq x_{1k} \leqq \sigma_k, \qquad 0 \leqq x_{k1} \leqq \rho_k \qquad (k \geqq 2). \tag{15}$$

This gives us the first row and the first column of the required matrix (x_{ij}). Since $\sum_{k=2}^{\infty} (\rho_k - x_{k1}) = \sum_{k=2}^{\infty} (\sigma_k - x_{1k}) = \infty$, we can now proceed inductively to the construction of all rows and columns.

Case 2. $s < \infty$.

In virtue of the lemma, we can choose numbers x_{1k}, x_{k1} $(k \geqq 2)$ which satisfy (14), (15), and $x_{1k} + x_{k1} \geqq \rho_k + \sigma_k + \rho_1 + \sigma_1 - s$ $(k \geqq 2)$. This gives us the first row and the first column of the required matrix (x_{ij}). Moreover

$$\sum_{k=2}^{\infty} (\rho_k - x_{k1}) = \sum_{k=2}^{\infty} (\sigma_k - x_{1k}) = s - \rho_1 - \sigma_1,$$

$$(\rho_k - x_{k1}) + (\sigma_k - x_{1k}) \leqq s - \rho_1 - \sigma_1 \qquad (k \geqq 2);$$

and, using the lemma repeatedly, we obtain an inductive procedure for constructing all rows and columns of the required matrix.

4. A (finite or infinite) square matrix is called *doubly stochastic* if its elements are nonnegative and if all row-sums and column-sums are equal to 1. Taking $\rho_k=\sigma_k=1$ $(k\geqq 1)$ in Theorem 1, we obtain the following result.

THEOREM 3. *The numbers $d_1, d_2, \cdots$ are the diagonal elements of some infinite doubly stochastic matrix if and only if*

$$0 \leqq d_k \leqq 1 \qquad (k \geqq 1), \tag{16}$$

$$2\left(1 - \inf_{j\geqq 1} d_j\right) \leqq \sum_{k=1}^{\infty} (1 - d_k) \leqq \infty. \tag{17}$$

Actually, a slightly stronger conclusion can be established by means of a more specialized argument. It can, in fact, be shown that if (16) and (17) are given, then $d_1, d_2, \cdots$ are the diagonal elements of a doubly stochastic matrix (d_{ik}) which is either the direct sum of finite doubly stochastic matrices or else satisfies $d_{ik}=0$ for $i\neq k$ and $\min(i, k)>m$, where m is a suitable number which depends on the d_k.*

Again, it is easy to see that any sequence $d_1, d_2, \cdots$ which satisfies (16) differs only slightly from the sequence of diagonal elements of some doubly stochastic matrix. To make this notion more precise, we define an *I-sequence* as any sequence $\{d_k\}$ which satisfies (16). Further, an *I*-sequence $\{d_k\}$ will be said to be a *D-sequence* if $d_1, d_2, \cdots$ are the diagonal elements of some doubly stochastic matrix. If $x=\{x_k\}$ is a bounded sequence, we define its norm by

$$\|x\| = \sup_{k\geqq 1} |x_k|. \tag{18}$$

THEOREM 4. *The set of D-sequences is everywhere dense, with respect to the norm* (18), *in the set of I-sequences.*

Let $x=\{x_k\}$ be any *I*-sequence and denote by ϵ any number satisfying $0<\epsilon<\frac{1}{2}$. We shall show that there exists a *D*-sequence $y=\{y_k\}$ such that $\|x-y\|<\epsilon$.

If $\sum_{k=1}^{\infty}(1-x_k)$ diverges, then, by Theorem 3, x is itself a *D*-sequence and we put $y_k=x_k$ $(k\geqq 1)$. If, on the other hand, the series converges, then $x_k\to 1$ $(k\to\infty)$ and so $x_k>\frac{1}{2}$ for $k>m$, say. If

$$y_k = \begin{cases} x_k & (k \leqq m), \\ x_k - \dfrac{\epsilon}{k} & (k > m), \end{cases}$$

then $\sum_{k=1}^{\infty}(1-y_k)$ diverges and therefore, by Theorem 3, $y=\{y_k\}$ is a *D*-sequence. This completes the proof.

5. Let $n\geqq 2$ be an integer. Taking $\rho_k=\sigma_k=1$ $(1\leqq k\leqq n)$ and $d_k=\rho_k=\sigma_k=0$ $(k>n)$ in Theorem 1, we obtain

* The two cases are not, of course, mutually exclusive.

THEOREM 5. *The numbers $d_1, \cdots, d_n$ are the diagonal elements of a doubly stochastic $n \times n$ matrix if and only if*

$$0 \leqq d_k \leqq 1 \qquad (1 \leqq k \leqq n), \tag{19}$$

$$2\left(1 - \min_{1 \leqq j \leqq n} d_j\right) \leqq \sum_{k=1}^{n} (1 - d_k). \tag{20}$$

This result was derived some years ago by A. Horn ([2], Theorem 9). We shall conclude our discussion by sketching yet a further proof depending on quite different ideas.

A vector $d = (d_1, \cdots, d_n)$ will be said to belong to the set $\mathfrak{D}$ if (19) and (20) are satisfied. We denote by $\mathfrak{E}$ the subset of $\mathfrak{D}$ consisting of those vectors all of whose components are 0 or 1. Thus a vector (of order n) belongs to $\mathfrak{E}$ if and only if all its components are 0 or 1 and the number of zero components is different from one.

It is a matter of routine verification that $\mathfrak{E}$ is the set of extreme points of $\mathfrak{D}$.† Hence, by a standard result (cf. [1], p. 24), any $d \in \mathfrak{D}$ can be represented in the form $d = t_1 e_1 + \cdots + t_r e_r$, where the t's are nonnegative numbers with sum 1 while the e's are vectors in $\mathfrak{E}$. But it is clear that e_k is the vector of diagonal elements of some $n \times n$ permutation matrix P_k. Hence d is the vector of diagonal elements of the doubly stochastic matrix $t_1 P_1 + \cdots + t_r P_r$. This establishes one half of Theorem 5. The converse inference is, of course, obvious.

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ON THE ANTICENTER OF A GROUP

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The concept of the anticenter $AC(G)$ of a group G was introduced by N. Levine [1]. In this paper we prove that the operation of forming the anticenter is idempotent, *i.e.*, $AC(G) = AC(AC(G))$. Using this the anticenter of all finitely generated abelian groups is determined. The anticenter is defined as follows:

DEFINITION 1. $R(G) = \{a \in G \mid ab = ba$ *implies there exists* $c \in G$ *such that* $a = c^j$, $b = c^k$ *for some integers* j *and* $k\}$.

DEFINITION 2. $AC(G) = \{a_1 a_2 \cdots a_n \mid a_i \in R(G)\}$.

We need the facts that $AC(G)$ is a normal subgroup of G, that $AC(G) = R(G)$ when G is abelian, and that for finite abelian groups $AC(G) = G$ if and only if G is cyclic [1].

† A point x of a convex set $\mathfrak{X}$ is said to be *extreme* if it cannot be represented in the form $x = \frac{1}{2}(x_1 + x_2)$, where $x_1, x_2 \in \mathfrak{X}$, $x_1 \neq x_2$.

THEOREM 1. $AC(AC(G)) = AC(G)$.

Proof. We shall prove that $R(G) \subset R(AC(G))$. If $a \in R(G)$, suppose $b \in AC(G)$ and $ab = ba$. Then there exists $c \in G$ such that $a = c^i$, $b = c^k$. We must show that there is an element d in $AC(G)$ with $a = d^u$, $b = d^v$. If s is the least positive integer for which c^s is in $AC(G)$, then $s \leqq j$ and $s \leqq k$. Moreover, s divides j and k, for otherwise $j = n + ms$ where $1 \leqq n \leqq s-1$. Then $c^j = c^n c^{ms}$, and since c^j and c^s belong to $AC(G)$, c^n would belong to $AC(G)$, $1 \leqq n \leqq s-1$, contrary to the definition of s. Similarly for k. Hence if $d = c^s$, we have $a = d^u$ and $b = d^v$, $d \in AC(G)$.

COROLLARY 1. *If G is finite and abelian then $AC(G) = R(G)$ is cyclic.*

This is an immediate consequence of [1], Theorems 3 and 4.

THEOREM 2. *If H is a subgroup of G, then $R(G) \cap H \subset R(H)$. If G is abelian, $AC(G) \cap H \subset AC(H)$.*

Proof. If $a \in R(G) \cap H$ and $b \in H$ such that $ab = ba$, then there exists $c \in G$ such that $a = c^i$, $b = c^k$. If s is the least positive integer such that $c^s \in H$, then, as in Theorem 1, $s \mid j$ and $s \mid k$, so $a = (c^s)^u$ and $b = (c^s)^v$ for some $c^s \in H$. The second statement follows from [1], Theorem 4.

THEOREM 3. *Let G be any finite abelian group with canonical decomposition as the direct product $G_1 \times \cdots \times G_z$ of cyclic groups G_i, where G_i has order $p_i^{e_i}$, $i = 1, \cdots, z$. If every prime p_i, $i = 1, \cdots, z$ divides the orders of at least two of the direct factors of G then*

$$AC(G) = R(G) = 1,$$

where 1 denotes the identity subgroup of G. If, on the other hand, $p_1, \cdots, p_y$ are those primes each of which divides the order of exactly one direct factor of G then

$$AC(G) = R(G) = G_1 \times \cdots \times G_y.$$

Proof. Suppose p is a prime associated with more than one direct factor. Let $H_1, \cdots, H_n$ be those direct factors having orders $p_1^{e_1}, \cdots, p_n^{e_n}$ respectively, $e_j > 0$. Then

$$R(G) \cap H_1 \times \cdots \times H_n \subset R(H_1 \times \cdots \times H_n).$$

We shall show that $R(H_1 \times \cdots \times H_n) = 1$. $R(H_1 \times \cdots \times H_n)$ is cyclic by Corollary 1. Let its generator be $r = a_1^{f_1} \cdots a_n^{f_n}$ where a_1 is the generator of H_1, a_2 that of H_2, etc. We shall prove that $r = 1$.

Since G is abelian, $ra = ar$, so there exists an element $a_1^{h_1} \cdots a_n^{h_n}$ such that

$$r = a_1^{f_1} \cdots a_n^{f_n} = (a_1^{h_1} \cdots a_n^{h_n})^j, \qquad a = (a_1^{h_1} \cdots a_n^{h_n})^k.$$

Hence, $h_1 k \equiv 1 \bmod p_1^{e_1}$, $h_2 k \equiv 0 \bmod p_2^{e_2}$, $\cdots$, $h_n \equiv 0 \bmod p_n^{e_n}$. Hence, $p_2^{e_2} \mid h_2, \cdots, p_n^{e_n} \mid h_n$. Then $p_2^{e_2} \mid f_2 \cdots$, $p_n^{e_n} \mid f_n$, so $r = a_1^{f_1}$. But, similarly, we can show that $p_1^{e_1} \mid f_1$. Hence $r = 1$, which proves the first assertion of the theorem.

Consider now $R(G)$ with its generator r_1. If, in the canonical representation of r_1 as powers of generators of $G_1, \cdots, G_z$, the generators of $H_1, H_2, \cdots$, or H_n occur with positive exponent, then by forming $(r_1)^g$ where g is the product of all $p_i^{\sigma_i}$ of primes different from p, we obtain a nontrivial element of $H_1 \times \cdots \times H_n$, belonging to $R(G)$. This is impossible. Hence, $R(G)$ is contained in the product of the G_i omitting $H_1, \cdots, H_n$.

By repeating this argument with any other prime which belongs to more than one direct factor G_i, we see that $R(G) \subset G_1 \times \cdots \times G_y$.

Let G_i be generated by a_i and have order $p_i^{\sigma_i}$, $i = 1, \cdots, z$. We shall prove that $a_i \in R(G)$ for $i = 1, \cdots, y$. This will prove that $R(G) = G_1 \times \cdots \times G_y$. By symmetry it suffices to show that $a_1 \in R(G)$.

For any element $a_1^{m_1} \cdots a_z^{m_z} \in G$ we must show the existence of an element $a_1^{\mu_1} \cdots a_z^{\mu_z} \in G$ and integers j, k such that

$$a_1 = (a_1^{\mu_1} \cdots a_z^{\mu_z})^j, \qquad a_1^{m_1} \cdots a_z^{m_z} = (a_1^{\mu_1} \cdots a_z^{\mu_z})^k.$$

This leads to equations

$$\begin{aligned}
&\mu_1 j \equiv 1 \bmod p_1^{\sigma_1}, \\
&\mu_2 j \equiv 0 \bmod p_2^{\sigma_2}, \\
&\quad\vdots \\
&\mu_z j \equiv 0 \bmod p_z^{\sigma_z}, \qquad (1) \\
&\mu_1 k \equiv m_1 \bmod p_1^{\sigma_1}, \qquad (2) \\
&\mu_2 k \equiv m_2 \bmod p_2^{\sigma_2}, \qquad (3) \\
&\quad\vdots \\
&\mu_z k \equiv m_z \bmod p_z^{\sigma_z}. \qquad (z+1)
\end{aligned}$$

Let $j = p_2^{\sigma_2} \cdots p_z^{\sigma_z}$. Since G_1 is the only direct factor of G whose order is a power of p_1, then $(j, p_1^{\sigma_1}) = 1$. Therefore there exist integers μ_1 and ν such that $1 = \mu_1 j + p_1^{\sigma_1} \nu$ and then the z equations of (1) are satisfied for arbitrary $\mu_2, \cdots, \mu_z$. Also

$$m_1 = \mu_1 m_1 j + p_1^{\sigma_1} \nu m_1,$$

so any integer $k \equiv m_1 j \bmod p_1^{\sigma_1}$ will satisfy (2). Let $k = m_1 j + p_1^{\sigma_1}$. Then $(k, p_i^{\sigma_i}) = 1$ for $2 \leqq i \leqq z$, so the equations (3), $\cdots$, $(z+1)$ can be solved for $\mu_2, \mu_3, \cdots, \mu_z$.

THEOREM 4. *If G is a finitely generated infinite abelian group which contains at least one nontrivial element of finite order then $R(G) = 1$.*

Proof. If $R(G)$ possesses a nontrivial element a of finite order and b is any element of infinite order, then $a = c^i$, $b = c^k$ implies that c has both finite and infinite order, which is impossible. In the same way, if $R(G)$ contains an element of infinite order and a is an element of G of finite order, the same argument holds.

THEOREM 5. *Let G be a finitely generated infinite abelian group without torsion. Then $R(G)=G$ if G is cyclic and $R(G)=1$ otherwise.*

Proof. If G is cyclic, $R(G)=G$ by [1], Theorem 2. If $a_1, a_2, \cdots, a_n$ constitute a basis of G, with $n>1$, suppose $R(G)$ contains the element $r=a_1^{m_1}\cdots a_n^{m_n}$. Then $a_1 r=r a_1$ implies that there exists $a_1^{\mu_1}\cdots a_n^{\mu_n}=c$ with $r=c^j$ and $a_1=c^k$. Then $\mu_1 k=1$, $\mu_2 k=\cdots=\mu_n^k=0$; hence $\mu_2=\cdots=\mu_n=0$, so $m_2=\cdots=m_n=0$. Similarly, $m_1=0$. Hence $r=1$ so $R(G)=1$.

Reference

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ON THE DEFINITENESS OF CERTAIN QUADRATIC FORMS ARISING IN A CONJECTURE OF L. J MORDELL

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H. S. Shapiro proposed the problem: Prove that

$$R_n[X_1,\cdots X_n]=\sum_1^n \frac{X_i}{X_{i+1}+X_{i+2}}\geqq\frac{n}{2}, \tag{1}$$

where the X_i are nonnegative numbers satisfying the cyclic relation $X_{n+k}=X_k$.

The relation has been proved for $n\leqq 6$. However, Zulauf* showed that (1) is false for $n\geqq 14$ and even. Mordell conjectured that (1) is false for $n\geqq 7$; however, the conjecture has not been proven. K. Goldberg of the National Bureau of Standards has tested (1) for $n=7$ for a large number of sets of values of the X_i, using a computing machine. These calculations failed to support Mordell's conjecture.

At the suggestion of S. Chowla, I have attempted to apply a generalization of Zulauf's argument to the cases $n=8$, 10, and 12, and a set of X_i falling in the neighborhoods of 0 and 1; that is, for $n=2s$ ($s=4$, 5 and 6) and $X_{2r-1}=1+a_r\epsilon$, $X_{2r}=b_r\epsilon$ for $\epsilon=$a small, positive number. In this case, R_n becomes $n/2$ plus a quadratic form (neglecting terms of higher order). The case treated by Zulauf produced a negative number in place of the generalized quadratic form found here. If the quadratic form were indefinite, Mordell's conjecture would have been supported. Unfortunately, the quadratic form turns out definite, so for this set of X_i, Mordell's conjecture is false.

With the above X_i, we have

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* Note on a conjecture of L. J. Mordell, Abh. Math. Sem. Univ. Hamburg, vol. 22, 1958, p. 240.

$$\frac{X_{2r+1}}{X_{2r}+X_{2r+1}}=\frac{1+a_r\epsilon}{1+[a_{r+1}+b_r]\epsilon}$$
$$=1+[a_r-a_{r+1}-b_r]\epsilon+[(a_{r+1}+b_r)^2-a_r(a_{r+1}+b_r)^2]\epsilon^2+\cdots,$$
$$\frac{X_{2r}}{X_{2r+1}+X_{2r+2}}=\frac{b_r\epsilon}{1+[a_{r+1}+b_{r+1}]\epsilon}$$
$$=b_r\epsilon-b_r[a_{r+1}+b_{r+1}]\epsilon^2+\cdots,$$

so that

$$\sum_1^n\frac{X_i}{X_{i+1}+X_{i+2}}=\frac{n}{2}+0\cdot\epsilon+K\epsilon^2+\cdots,$$

where

$$K=K[a_1,\cdots,a_s;b_1,\cdots,b_s]$$
$$=\sum_1^s a_i^2+\sum_1^s b_i^2-\sum_1^s a_ia_{i+1}+\sum_1^s b_ia_{i+1}-\sum_1^s b_ib_{i+1}-\sum_1^s a_ib_i.$$

Thus for the cases $n=8$, 10, 12 we have explicitly:

$n=8$:

$$K=[a_1-\tfrac{1}{2}(a_2+a_4+b_1-b_4)]^2+\tfrac{3}{4}[a_2-\tfrac{1}{3}(2a_3+a_4-b_1+2b_2-b_4)]^2$$
$$+\tfrac{2}{3}[a_3-a_4+\tfrac{1}{4}(b_1+b_2-3b_3+b_4)]^2$$
$$+\tfrac{5}{8}[b_1-\tfrac{1}{5}(3b_2-b_3+3b_4)]^2+\tfrac{2}{5}[b_2-\tfrac{3}{4}b_3-\tfrac{1}{4}b_4]^2+\tfrac{3}{8}[b_3-b_4]^2.$$

$n=10$:

$$K=[a_1-\tfrac{1}{2}(a_2+a_5+b_1+b_5)]^2+\tfrac{3}{4}[a_2-\tfrac{1}{3}(2a_3+a_5-b_1+2b_2-b_5)]^2$$
$$+\tfrac{2}{3}[a_3-\tfrac{1}{4}(3a_4+a_5-b_1-b_2+3b_3-b_5)]^2$$
$$+\tfrac{5}{8}[a_4-a_5+\tfrac{1}{5}(b_1+b_2+b_3-4b_4+b_5)]^2$$
$$+\tfrac{3}{5}[b_1-\tfrac{1}{6}(4b_2-b_3-b_4+2b_5)]^2+\tfrac{1}{3}[b_2-b_3+\tfrac{1}{2}(b_4-b_5)]^2$$
$$+\tfrac{1}{2}[b_3-b_4]^2+\tfrac{1}{2}[b_4-b_5]^2.$$

$n=12$:

$$K=[a_1-\tfrac{1}{2}(a_2+a_6+b_1-b_6)]^2+\tfrac{3}{4}[a_2-\tfrac{1}{3}(2a_3+a_6-b_1+2b_2-b_6)]^2$$
$$+\tfrac{2}{3}[a_3-\tfrac{1}{4}(3a_4+a_6-b_1-b_2+3b_3-b_6)]^2$$
$$+\tfrac{5}{8}[a_4-\tfrac{1}{5}(4a_5+a_6-b_1-b_2-b_3+4b_4-b_6)]^2$$
$$+\tfrac{3}{5}[a_5-a_6+\tfrac{1}{6}(b_1+b_2+b_3+b_4+5b_5-b_6)]^2$$
$$+\tfrac{7}{12}[b_1-\tfrac{1}{7}(5b_2-b_3-b_4-b_5+5b_6)]^2$$
$$+\tfrac{4}{7}[b_2-\tfrac{1}{4}(5b_3-2b_4-2b_5+3b_6)]^2+\tfrac{1}{3}[b_3-2b_4+2b_5-b_6]^2.$$

ON THE COMMUTIVITY OF THE CLOSURE AND INTERIOR OPERATORS IN TOPOLOGICAL SPACES

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If E is the set of rationals in the space of reals R, then $c \operatorname{Int} E = \phi$ and $\operatorname{Int} cE = R$ where c denotes the closure operator and Int denotes the interior operator. It is thus evident that the closure and interior operators do not permute.

DEFINITION. *A set E in a topological space X has property Q if and only if* $c \operatorname{Int} E = \operatorname{Int} cE$.

It is the intent of this paper to give a characterization of property Q in a general topological space and, as a corollary, a characterization in a connected space.

LEMMA 1. *For E a subset of a topological space X, E has property Q if and only if $\mathcal{C}E$ has property Q. ($\mathcal{C}$ denotes the complement operator.)*

Proof. Necessity. Let E have property Q. Then $\operatorname{Int} c\mathcal{C}E = \mathcal{C}c\mathcal{C}c\mathcal{C}E = \mathcal{C}c \operatorname{Int} E = \mathcal{C} \operatorname{Int} cE = \mathcal{C} \operatorname{Int} c\mathcal{C}\mathcal{C}E = \mathcal{C}\mathcal{C}c\mathcal{C}c\mathcal{C}\mathcal{C}E = c \operatorname{Int} \mathcal{C}E$. *Sufficiency.* Let $\mathcal{C}E$ have property Q. Then $E = \mathcal{C}\mathcal{C}E$ and thus, by the above, E has property Q.

LEMMA 2. *For E a subset of a topological space X, E has property Q implies that cE has property Q.*

Proof. $c \operatorname{Int} cE = c \operatorname{Int} c\mathcal{C}\mathcal{C}E = c\mathcal{C}c\mathcal{C}c\mathcal{C}\mathcal{C}E = c\mathcal{C}c \operatorname{Int} \mathcal{C}E = c\mathcal{C} \operatorname{Int} c\mathcal{C}E$ (by Lemma 1) $= c\mathcal{C}\mathcal{C}c\mathcal{C}c\mathcal{C}\mathcal{C}E = c \operatorname{Int} E = \operatorname{Int} cE$ (since E has property Q) $= \operatorname{Int} ccE$.

The converse of Lemma 2 is false. For if E is the set of rationals in the reals R, then E does not have property Q, but $cE = R$ and R does have property Q.

LEMMA 3. *For E a subset of a topological space X, E has property Q implies that* $\operatorname{Int} E$ *has property Q.*

Proof. E has property Q and thus $\operatorname{Int} E = \mathcal{C}c\mathcal{C}E$ has property Q by Lemmas 1 and 2.

The converse of Lemma 3 is false. For let E be the set of rationals in the reals R. Then E does not have property Q, but $\operatorname{Int} E = \phi$ and ϕ does have property Q.

LEMMA 4. *Let E be a subset of a topological space X. If E or $\mathcal{C}E$ is nowhere dense, then E has property Q.*

Proof. Let E be nowhere dense in X. Then $\operatorname{Int} cE = \phi$. But $\operatorname{Int} E \subset \operatorname{Int} cE = \phi$ and thus $\operatorname{Int} E = \phi$. Hence $c \operatorname{Int} E = \phi$ and E has property Q. If $\mathcal{C}E$ is nowhere dense, then $\mathcal{C}E$ has property Q and by Lemma 1, E has property Q.

The converse of Lemma 4 is false. Let $X: a, b$ and let the open sets be ϕ, (a), (b), and X. Then (a) has property Q since (a) is both open and closed, but (a) is not nowhere dense nor is $\mathcal{C}(a)$ nowhere dense in X.

LEMMA 5. *Let A be closed in X and $B \subset A$. Then $cB = c_A B$ where c_A is the closure operator in the subspace A.*

Proof. Since $B \subset A$, we have $cB \subset cA = A$. But $c_A B = A \cap cB = cB$.

LEMMA 6. *Let A be open in X and $B \subset A$. Then* $\operatorname{Int} B = \operatorname{Int}_A B$ *where* Int_A *denotes the interior operator in the subspace A.*

Proof. Let $x \in \operatorname{Int} B$. Then $x \in G \subset B$ where G is open in X. It follows that $x \in G \cap A \subset B$ and thus $x \in \operatorname{Int}_A B$. Conversely, let $x \in \operatorname{Int}_A B$. Then $x \in O \cap A \subset B$ where O is open in X. Then $x \in \operatorname{Int} B$ since $O \cap A$ is open in X.

LEMMA 7. *Let A be open and closed in a topological space X. Let $B \subset A$ and $C \subset \mathcal{C}A$. Then* $\operatorname{Int}(B \cup C) = \operatorname{Int} B \cup \operatorname{Int} C$.

Proof. $\operatorname{Int} B \subset \operatorname{Int}(B \cup C)$ and $\operatorname{Int} C \subset \operatorname{Int}(B \cup C)$. Thus $\operatorname{Int} B \cup \operatorname{Int} C \subset \operatorname{Int}(B \cup C)$. Now let $x \in \operatorname{Int}(B \cup C)$. Then $x \in G \subset B \cup C$ where G is open in X. Case 1. $x \in A$. Then $x \in G \cap A \subset (B \cup C) \cap A = B$. Thus $x \in \operatorname{Int} B$. Case 2. $x \in \mathcal{C}A$. Then $x \in G \cap \mathcal{C}A \subset (B \cup C) \cap \mathcal{C}A = C$. Then $x \in \operatorname{Int} C$.

LEMMA 8. *Let A be open and closed in a topological space X and P a nowhere dense set in X. Then $P \cap A$ is nowhere dense in A.*

Proof. $\operatorname{Int}_A c_A P \cap A = \operatorname{Int} cP \cap A$ (by Lemmas 5 and 6) $\subset \operatorname{Int} cP = \phi$ since P is nowhere dense.

We now give a characterization of property Q in a general topological space.

THEOREM 1.* *Let E be a subset of a topological space X. Then E has property Q if and only if $E = (A - P) \cup (P - A)$ where A is both open and closed and P is nowhere dense.*

Proof. Sufficiency. Let $E = (A - P) \cup (P - A)$ where A is open and closed and P is nowhere dense. Now $A - P \subset A$ and $P - A \subset \mathcal{C}A$. Hence

$$\begin{aligned} c \operatorname{Int} E &= c \operatorname{Int} \{(A - P) \cup (P - A)\} \\ &= c\{\operatorname{Int} (A - P) \cup \operatorname{Int} (P - A)\} && \text{(by Lemma 7)} \\ &= c\{\operatorname{Int}_A (A - P) \cup \operatorname{Int}_{\mathcal{C}A}(P - A)\} && \text{(by Lemma 6)} \\ &= c \operatorname{Int}_A (A - P) \cup c \operatorname{Int}_{\mathcal{C}A} (P - A) \\ &= c_A \operatorname{Int}_A (A - P) \cup c\mathcal{C}_A \operatorname{Int}_{\mathcal{C}A} (P - A) && \text{(by Lemma 5).} \end{aligned}$$

Similarly $\operatorname{Int} cE = \operatorname{Int}_A c_A(A - P) \cup \operatorname{Int}_{\mathcal{C}A} c\mathcal{C}_A(P - A)$. Now $A - P = A - A \cap P$ and $P \cap A$ is nowhere dense in A by Lemma 8. Then $A - P$ is the complement relative to A of a set nowhere dense in A and thus $c_A \operatorname{Int}_A(A - P) = \operatorname{Int}_A c_A(A - P)$ by Lemma 4. Moreover $P - A = P \cap \mathcal{C}A$ is nowhere dense in $\mathcal{C}A$ by Lemma 8. Thus $c\mathcal{C}_A \operatorname{Int}_{\mathcal{C}A}(P - A) = \operatorname{Int}_{\mathcal{C}A} c\mathcal{C}_A(P - A)$ by Lemma 4 and it now follows that $c \operatorname{Int} E = \operatorname{Int} cE$.

* The author is indebted to the referee for conjecturing this theorem.

Necessity. Suppose E has property Q. Then let $A = c\ \mathrm{Int}\ E = \mathrm{Int}\ cE$. A then is both open and closed in X. We now show that $E-A$ and $A-E$ are each nowhere dense. $\mathrm{Int}\ c(E-A) \subset \mathrm{Int}\ c(E) = A$, but also $\mathrm{Int}\ c(E-A) \subset \mathrm{Int}\ c(\mathcal{C}A) = \mathcal{C}A$. Hence $\mathrm{Int}\ c(E-A) = \phi$. In addition, $\mathrm{Int}\ c(A-E) \subset \mathrm{Int}\ cA = A$ and also $\mathrm{Int}\ c(A-E) \subset \mathrm{Int}\ c\mathcal{C}E = \mathcal{C}c\mathcal{C}c\mathcal{C}E = \mathcal{C}c\ \mathrm{Int}\ E = \mathcal{C}A$. Hence $\mathrm{Int}\ c(A-E) = \phi$. Let $P = (A-E) \cup (E-A)$. Then P is nowhere dense and the proof will be complete when we show that $E = (A-P) \cup (P-A)$.

To this end we observe that

$$(A-P) \cup (P-A) = (A \cap \mathcal{C}P) \cup (P \cap \mathcal{C}A)$$
$$= [A \cap \mathcal{C}\{(A \cap \mathcal{C}E) \cup (E \cap \mathcal{C}A)\}] \cup [\{(A \cap \mathcal{C}E) \cup (E \cap \mathcal{C}A)\} \cap \mathcal{C}A]$$
$$= [A \cap \mathcal{C}(A \cap \mathcal{C}E) \cap \mathcal{C}(E \cap \mathcal{C}A)] \cup [E \cap \mathcal{C}A] = [A \cap (\mathcal{C}A \cup E) \cap (\mathcal{C}E \cup A)] \cup [E \cap \mathcal{C}A]$$
$$= [(A \cap E) \cap (\mathcal{C}E \cup A)] \cup [E \cap \mathcal{C}A] = (A \cap E) \cup (E \cap \mathcal{C}A) = E.$$

COROLLARY. *Let X be a connected topological space and E a subset of X. Then E has property Q if and only if E is nowhere dense or $\mathcal{C}E$ is nowhere dense.*

Proof. By the above theorem, E has property Q if and only if E can be written as $(A-P) \cup (P-A)$ where A is both open and closed and P is nowhere dense. Since X is connected, the only sets which are both open and closed in X are X itself and ϕ. Thus E has property Q if and only if $E = (\phi - P) \cup (P - \phi)$ or $E = (X-P) \cup (P-X)$. In the former case E is nowhere dense and in the latter case $\mathcal{C}E$ is nowhere dense.

LEMMA 9. *Let E, F, and G be any sets. Let Δ be the symmetric difference operator. Then* (1) $E \Delta F = F \Delta E$, (2) $(E \Delta F) \Delta G = E \Delta (F \Delta G)$ *and* (3) $E \cap (F \Delta G) = (E \cap F) \Delta (E \cap G)$.

The results in the above lemma are well known and proofs will not be given.

THEOREM 2. *Let E and E^* have property Q in a topological space X. Then $E \cap E^*$ has property Q.*

Proof. $E = A \Delta P$, and $E^* = A^* \Delta P^*$ where A and A^* are both open and closed and P and P^* are both nowhere dense by Theorem 1. Then

$$\begin{aligned} E \cap E^* &= E \cap (A^* \Delta P^*) \\ &= (E \cap A^*) \Delta (E \cap P^*) \text{ (by Lemma 9)} \\ &= ((A \cap A^*) \Delta (P \cap A^*)) \Delta ((A \cap P^*) \Delta (P \cap P^*)) \text{ (by Lemma 9)} \\ &= (A \cap A^*) \Delta \{(P \cap A^*) \Delta (A \cap P^*) \Delta (P \cap P^*)\} \text{ (by Lemma 9)} \\ &= A^{**} \Delta P^{**}, \end{aligned}$$

where $A^{**} = A \cap A^*$ and $P^{**} = (P \cap A^*) \Delta (A \cap P^*) \Delta (P \cap P^*)$. But A^{**} is both open and closed and P^{**} is nowhere dense. Thus $E \cap E^*$ has property Q by Theorem 1.

COROLLARY. *Let E and E^* have property Q in a topological space X. Then* (1) $E \cup E^*$ *has property* Q, (2) $E - E^*$ *has property* Q *and* (3) fr E *has property* Q *where* fr *denotes the frontier operator.*

Proof. $E \cup E^* = \mathfrak{c}\mathfrak{c}(E \cup E^*) = \mathfrak{c}\{\mathfrak{c}E \cap \mathfrak{c}E^*\}$. But $\mathfrak{c}E$ and $\mathfrak{c}E^*$ have property Q and thus $\mathfrak{c}E \cap \mathfrak{c}E^*$ has property Q by the above theorem. Then $E \cup E^*$ has property Q by Lemma 1. (2) follows from the identity $E - E^* = E \cap \mathfrak{c}E^*$ and Lemma 1 and Theorem 2. To show (3) we note that fr $E = cE \cap c\mathfrak{c}E$ and cE has property Q as well as $c\mathfrak{c}E$ by Lemmas 1 and 2. Thus fr E has property Q by Theorem 2.

Reference

J. L. Kelley, General Topology, New York, 1955.

NOTE ON A PAPER OF KLAMKIN CONCERNING STIRLING NUMBERS*

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M. S. Klamkin [5] found the closed summation formula

$$K_n = \sum_{k=1}^{\infty} k^n x^k = (1-x)^{-n-1} \sum_{k=0}^{n} x^{k+1}(1-x)^{n-k}\Delta^k 1^n, \tag{1}$$

and remarked that the following relations allow K_n to then be expressed in terms of generalized Bernoulli numbers or Stirling numbers of the *first* kind:

$$\Delta^k 0^n + \Delta^{k+1} 0^n = \Delta^k 1^n, \tag{2}$$

$$\Delta^k 0^n = \frac{n!}{(n-k)!} B_{n-k}^{(-k)}, \tag{3}$$

$$S_n^m = \binom{n-1}{m-1} B_{n-m}^{(n)}, \tag{4}$$

where S_n^m is a Stirling number of the first kind as defined in [4].

There is considerable difficulty in passing from (3) to (4) however. This was also detected by Paasche [6]. From (4) we would have to write

$$B_{n-k}^{(-k)} = \frac{S_{-k}^{-n}}{\binom{-k-1}{-n-1}}, \tag{5}$$

which appears to be meaningless because S_b^a has no interpretation for both a and b negative, and there is a similar difficulty for the binomial coefficient.

If we look a bit deeper we may, however, show that the passage from (3) to (4) leads to no formula expressing $\Delta^k 0^n$ in terms of S_n^m but a mere tautology.

* Presented to the Allegheny Mountain Section of the Association, Pittsburgh, May 2, 1959. See this MONTHLY, vol. 66, 1959, p. 640.

To see this clearly, we note first some facts about Stirling numbers. In another standard combinatorial notation [2] the Stirling numbers of first kind, S_1, and of second kind, S_2, are related to those defined by Jordan in the following manner:

$$S_n^k = (-1)^{n-k} S_1(n-1, n-k), \tag{6}$$

and

$$\mathfrak{S}_n^k = S_2(k, n-k) = \frac{1}{k!}\Delta^k 0^n, \tag{7}$$

the latter part of (7) being of course the well-known fact that Stirling numbers of the second kind are nothing but higher differences of zero.

Now [2] it is possible to make sense of S_{-k}^{-n} by use of (6) and the further information that

$$S_1(-n-1, k) = S_2(n, k), \qquad S_2(-n-1, k) = S_1(n, k). \tag{8}$$

Indeed we find by substitutions that

$$\begin{aligned} S_{-k}^{-n} &= (-1)^{-n+k} S_1(-k-1, -k+n) = (-1)^{n-k} S_2(k, n-k) \\ &= (-1)^{n-k}\Delta^k 0^n. \end{aligned} \tag{9}$$

We may also find an interpretation for the binomial coefficient in (5) if we make use of the properties

$$\binom{x}{y} = \binom{x}{x-y} \quad \text{and} \quad \binom{-x}{y} = (-1)^y \binom{x+y-1}{y}. \tag{10}$$

Manipulative substitutions with these show that either

$$\binom{-k-1}{-n-1} = (-1)^{n-k}\binom{n}{k} \quad \text{or} \quad (-1)^{n-k-1}\binom{n}{k}.$$

Using this and (9) in (5) to simplify (3) we obtain

$$\Delta^k 0^n = \frac{n!}{(n-k)!} \cdot \frac{\pm\,\Delta^k 0^n}{\binom{n}{k} k!} = \pm\,\Delta^k 0^n,$$

and, choosing the plus sign, we see that the passage is indeed a tautology.

Now, it *is* possible to express the Stirling numbers of the second kind in terms of the Stirling numbers of the first kind as the author [2] has shown. The converse problem was first solved by Schlömilch, as pointed out in [2]. The formulas given in [2] then allow a solution to the present paradox and afford a method of expressing K_n here in terms of either S_1 or S_2. The existence of formulas relating S_1 to S_2 or conversely as shown in [2] depends on the fact that

the functions $\log(1+x)$ and e^x-1 are inverses of one another. The writer is indebted to the referee for pointing out that Jabotinski [3] has given general theorems which relate the coefficients in two inverse functions.

The writer would like to indicate here that after writing [2] he found that Goldberg [1] had also found a solution to the problem of expressing the Stirling numbers in terms of each other.

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CONCAVE FUNCTIONALS AND A PROBLEM OF BELLMAN

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In [1], p. 42, Bellman poses the problems of maximizing functions of real variables of the form

$$F(x_1, \cdots, x_n) = \sum_{i=1}^{n} \phi(x_i) \tag{1}$$

and

$$F(x_1, \cdots, x_n; y_1, \cdots, y_n) = \sum_{i=1}^{n} \phi(x_i, y_i),$$

subject to

$$x_i, y_i \geqq 0, \qquad \sum_{i=1}^{m} x_i = C_1, \qquad \sum_{i=1}^{n} y_i = C_2,$$

where the function ϕ is strictly concave and monotone increasing in each of its arguments. In this note the problem is stated for real concave functionals on a partially ordered vector space and we show that the maximum occurs when the vectors x_i are all equal.

Let X be a partially ordered real linear vector space ([2], pp. 10–16), X^+ the positive cone in X and ϕ a real-valued functional on X^+ which is *concave* there. For any C in X^+, let R_n be the set of n-tuples of vectors x_i in X^+ such that $x_1+x_2+\cdots+x_n=C$, let F be as in (1), and set

$$f_n(C) = \max_{R_n} F(x_1, \cdots, x_n).$$

Then we will see that $f_n(C) = n\phi(1/nC)$. For we notice that $f_1(c) = \phi(C)$ while

$$f_n(C) = \max_{0 \leq y \leq C} \{\phi(y) + f_{n-1}(C - y)\}, \qquad n \geq 2, \tag{2}$$

and, from the concavity of ϕ, we see that

$$\frac{1}{n}\phi(y) + \frac{n-1}{n}\phi\left(\frac{1}{n-1}(C - y)\right) \leq \phi\left(\frac{1}{n}C\right).$$

The induction argument is now quite clear and the claim is proven. From this last line we note that the maximum is unique if ϕ is *strictly concave.*

If ϕ is monotone nondecreasing, we may modify the definition of R_n by requiring only $x_1 + \cdots + x_n \leq C$. Equation (2) can still be established easily, and the inductive proof is still valid.

This solves Bellman's problems, and leads to a number of interesting and sometimes familiar corollaries. For example, if X is the space of real continuous functions on a closed interval $[a, b]$, then it is easy to verify that the functional $\phi(x) = \min_{[a,b]} x(t)$ is concave and monotone nondecreasing with respect to the natural ordering $x(t) \geq y(t)$ if $x(t) - y(t) \geq 0$, $a \leq t \leq b$. Here R_n is the set of nonnegative functions $x_i(t)$ with $x_1(t) + x_2(t) + \cdots + x_n(t) \leq C(t)$ for a fixed nonnegative function $C(t)$ and we conclude that

$$\max_{R_n} \sum_{i=1}^{n} \min_{[a,b]} (x_i(t)) = \min_{[a,b]} C(t).$$

If we take X to be a normed linear space and $\phi(x) = -\|x\|$, we obtain $\min \sum_{i=1}^{n} \|x_i\| = \|C\|$ whenever C, x_i are nonnegative vectors such that $x_1 + \cdots + x_n = C$; that is, the straight line is the shortest distance between two points.

Lastly, let x be a real variable, $\phi(x) = -x \log x$, and R_n the n-tuples $x_i \geq 0$, $x_1 + \cdots + x_n = 1$. We conclude from the strict concavity of ϕ that

$$\max_{R_n} \left(-\sum_{i=1}^{n} x_i \log x_i\right) = \log n$$

and that the maximum is unique. This, in information theory, is the well-known assertion that the maximum information content of a source having n-elements is $\log n$, and is achieved only when all elements have equal probability* ([3], p. 15).

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* This application was pointed out to me by H. Young.

ON EXTREMALS OF FINITE SUMS AND DEFINITE INTEGRALS

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1. Introduction. In many physical problems it is desirable to optimize a functional involving finite differences of the unknown function. Such problems often arise when the functional is a definite integral and the independent variable is time. The unknown function is thus involved in the functional at various delayed times. The problem parallels the problem of the calculus of variations in which the functional involves derivatives rather than differences. In the case when the functional is a finite sum involving finite differences of the unknown function, others [1] have shown that a necessary condition takes the form of a difference equation which is analogous to Euler's equation in the calculus of variations. The present note extends the method to functionals which are finite sums involving differences higher than the first and to functionals which are infinite sums, or definite integrals, involving finite differences of the unknown functions.

2. Finite sum involving a first difference. Let $\Delta y(i) \equiv y(i+k) - y(i)$, where k is a fixed integer and i is a variable integer such that $0 \leqq i \leqq n$. Assume that $F(i, y(i), \Delta y(i))$ possesses continuous second derivatives with respect to i, y, and Δy in the region defined by $0 \leqq i \leqq n$. Let it be required that $y(i)$ be a function such that the finite sum $I = \sum_{i=0}^{n} F(i, y, \Delta y)$ be an extremum.

A necessary condition is that $y(i)$ satisfy

$$\frac{\partial I}{\partial y(i)} = F_y(i) - F_{\Delta y}(i) + F_{\Delta y}(i-k) = 0,$$

or $F_y(i) - \Delta F_{\Delta y}(i-k) = 0$, which, in general, is a second-order difference equation, and in which $F(i) = 0$ for $i < 0$ and $i > n+k$.

3. Finite sum involving first and second differences. If the finite sum integrand is a function of second differences, $\Delta^2 y(i) \equiv \Delta y(i+k) - \Delta y(i)$, as well as first, or $I = \sum_{i=0}^{n} F(i, y, \Delta y, \Delta^2 y)$ then, for an extremum, $y(i)$ must satisfy

$$\frac{\partial I}{\partial y(i)} = F_y(i) - F_{\Delta y}(i) + F_{\Delta y}(i-k) + F_{\Delta^2 y}(i) - 2F_{\Delta^2 y}(i-k) + F_{\Delta^2 y}(i-2k) = 0,$$

or $F_y(i) - \Delta F_{\Delta y}(i-k) + \Delta^2 F_{\Delta^2 y}(i-2k) = 0$, which, in general, is a fourth-order difference equation, and in which $F(i) = 0$ for $i < 0$ and $i > n+2k$.

4. Finite sum involving higher differences. By induction on the above results it can be shown that if $I = \sum_{i=0}^{n} F(i, y, \Delta y, \cdots, \Delta^r y)$ is an extremum then $y(i)$ must satisfy the difference equation

$$\sum_{j=0}^{r} (-1)^j \Delta^j F_{\Delta^j y}(i - jk) = 0,$$

where $\Delta^0 \equiv 1$, which, in general, is of order $2r$, and in which $F(i) = 0$ for $i < 0$ and $i > n + rk$.

5. Definite integral involving a first difference. Assume

$$I = \int_a^b F(x, y, \Delta y)dx, \tag{1}$$

where $\Delta y \equiv y(x+\tau) - y(x)$, possesses continuous second derivatives with respect to x, y, and Δy in the region (a, b).

We note that

$$\begin{aligned} I = &\sum_{i=0}^{n-1} \int_{R_i} F(x(i), y(x(i)), \Delta y(x(i)))d(x(i)) \\ &+ \int_{a+n}^{b} F(x(n), y(x(n)), \Delta y(x(n)))d(x(n)), \end{aligned} \tag{2}$$

where

$$x(i) = \begin{cases} a + i\tau + \xi, & 0 \leqq \xi < \tau, \\ 0, & \xi < 0, \xi \geqq \tau, \end{cases}$$

and

$$R_i: a + i\tau \leqq x(i) < a + (i + 1)\tau.$$

If $y(x)$ makes the integral in (1) stationary it must also make the sum of the integrals in (2) stationary.

By considering each $x(i)$ as an independent variable, a necessary condition that the $y(x(i))$ make (2) stationary is that the $y(x(i))$ satisfy the equations

$$\frac{\partial I}{\partial y(x(i))} = F_y(x(i)) - F_{\Delta y}(x(i)) + F_{\Delta y}(x(i) - \tau) = 0,$$

or

$$F_y(x(i)) - F_{\Delta y}(x(i) - \tau) = 0. \tag{3}$$

Since $x = x(i)$ for $a + i\tau \leqq x < a + (i+1)\tau$, then (3) may be written simply as

$$F_y(x) - F_{\Delta y}(x - \tau) = 0. \tag{4}$$

Thus, if $y(x)$ makes the integral in (1) an extremum it must satisfy (4). It is observed that in (4), $F(x) = 0$ for $x < a$ and $x > b + \tau$.

6. Definite integral involving higher differences. By methods similar to those above it can be shown that for the integral

$$\int_a^b F(x, y, \Delta y, \cdots, \Delta^r y)dx$$

to be an extremum a necessary condition is that the function $y(x)$ satisfy the difference equation

$$\sum_{j=0}^{r} (-1)^j \Delta^j F_{\Delta^j y}(x - j\tau) = 0, \tag{5}$$

where $\Delta^0 \equiv 1$ and in which $F(x) = 0$ for $x < a$ and $x > b + r\tau$.

7. Discussion. The solutions of the above difference equations make the corresponding functionals stationary, but they are not necessarily extrema. However, in most physical problems the existence of an extremum can be intuitively recognized. If there exist several solutions to the difference equation, it is relatively easy to verify which makes the corresponding functional an extremum.

The boundary conditions are usually specified at the end points and usually consist of values of the unknown function and/or values of first and/or higher differences of the function. In many problems, however, boundary conditions may be specified within the interval in the same manner.

If we define $E^h f(x) \equiv f(x+h\tau)$, we may write (5) as

$$\sum_{j=0}^{r} (-1)^j (\Delta E^{-1})^j F_{\Delta^j y}(x) = 0.$$

Euler's equation for the extremals of the corresponding functional involving derivatives is

$$\sum_{j=0}^{r} (-1)^j D_x^j F_{y^{(j)}}(x) = 0.$$

Thus, we see that the equations are analogous with the operator ΔE^{-1} corresponding to the operator D_x.

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THE SILOV BOUNDARY FOR A LINEAR SPACE OF CONTINUOUS FUNCTIONS

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The theorem of Silov (see [4], p. 80) states that if A is an algebra of real or complex valued continuous functions on a compact Hausdorff space X, then there is a smallest closed subset F of X on which each function in A assumes its maximum modulus. The set F is called the Silov boundary of A in X. Arens and Singer ([1], Theorem 2.4) have shown the existence of such a boundary for a subfamily A of continuous functions on a topological space X provided that A is a multiplicative semigroup, the functions in A vanish at infinity, and X has a

* The author wishes to acknowledge financial support from the National Science Foundation.

basis of sets of the form $\{x: |f_i(x)| < \epsilon, i=1, \cdots, n\}$ for functions $f_1, \cdots, f_n$ in A. Recently Bauer [2], and Bishop and de Leeuw [3] have shown the existence of more refined (nonclosed) boundaries for linear spaces of continuous functions on a compact space. The proofs of [2] and [3] use a fair amount of machinery from the theory of linear topological spaces. Our purpose here is to give an elementary proof of the existence of a closed boundary for linear subspaces of $C(X)$. The theorem can be deduced from that of Arens and Singer by taking exponentials, but the present proof is simpler in view of the topological requirement of the Arens-Singer theorem.

Let $C(X)$ denote the space of all continuous real- or complex-valued continuous functions on the compact Hausdorff space X. Let $\|f\|$ denote the maximum modulus of f on X. The weak topology on X induced by a family $\mathfrak{F}$ of functions on X has as a basis at x_0 the sets

$$\{x: |f_i(x) - f_i(x_0)| < \epsilon, i = 1, \cdots, n\}, \tag{1}$$

where $f_1, \cdots, f_n$ is any finite set of functions in $\mathfrak{F}$. Recall ([4] Theorem 5G, p. 12) that if $\mathfrak{F} \subset C(X)$, and $\mathfrak{F}$ contains the constant functions and separates points of X (*i.e.*, if $x \neq y$, there is some $f \in \mathfrak{F}$ such that $f(x) \neq f(y)$), then the weak topology induced by $\mathfrak{F}$ coincides with the given compact topology of X.

THEOREM. *If X is compact Hausdorff, and H is a point-separating linear subspace of $C(X)$, containing the constant functions, then there is a (obviously unique) closed set F such that*

(i) *for each $f \in H$, there is some $x \in F$ such that $|f(x)| = \|f\|$, and*

(ii) *if B is any closed set with property* (i) *then $F \subset B$.*

Proof. Let $\mathfrak{N}$ be a maximal nest of closed sets having property (i), and let $F = \bigcap \mathfrak{N}$. Clearly F is closed. By the definition of $\mathfrak{N}$, $M(f) = \{x: |f(x)| = \|f\|\}$ intersects each set in $\mathfrak{N}$, and since $\mathfrak{N}$ is a nest, $M(f)$ intersects any finite number of sets in $\mathfrak{N}$. By the finite intersection property of the compact space X, $M(f)$ intersects F, which says that F has property (i).

Now suppose that B is a closed set with property (i), and that $x_0 \in F - B$. Let V be a neighborhood of the form (1) such that $x_0 \in V \subset X - B$. Since H is a linear space containing the constants, the functions $g_i = f_i - f_i(x_0)$ are in H and

$$V = \{x: |g_i(x)| < \epsilon, i = 1, \cdots, n\}. \tag{2}$$

Since $F = \bigcap \mathfrak{N}$, where $\mathfrak{N}$ is a maximal nest, the set $F - V$ does not have property (i), and there is a function $f \in H$ such that $\|f\| > \sup\{|f(x)|: x \in F - V\} = \|f\|_{F-V}$. Let $k = mf$, where m is so large that $\|k\| - \|k\|_{F-V} > \|g_1\| + \cdots + \|g_n\|$. If $x \in V$ and α is any complex number with $|\alpha| = 1$, then $|k(x) + \alpha g_i(x)| < \|k\| + \epsilon$. If $x \in F - V$, then $|k(x) + \alpha g_i(x)| < \|k\|_{F-V} + \|g_i\| < \|k\|$. Therefore, for any i, any α with $|\alpha| = 1$, and any $x \in F$, $|k(x) + \alpha g_i(x)| < \|k\| + \epsilon$. Since F has property (i),

$$|k(x) + \alpha g_i(x)| < \|k\| + \epsilon \tag{3}$$

for all $x \in X$. Let t be any point of X such that $\|k\| = |k(t)|$. Pick α so that

$|k(t)+\alpha g_i(t)| = \|k\| + |g_i(t)|$. From (3) we conclude that $|g_i(t)| < \epsilon$. Since g_i can be any of the functions occurring in (2), it follows that $t \in V$. That is, k only assumes its maximum on V, and hence not on B, which contradicts the assumption that B has property (i). Therefore, if B has property (i), $F \subset B$ as claimed.

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AN ELEMENTARY PROOF OF THE FORMULA $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$

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The formula in the title has been well known, but its various known proofs are less elementary, (see for example, [1], p. 219, 360, [3], p. 237, 267, 324, [4], Problem 99, p. 196, [5], p. 379). The following proof is quite elementary in character.

For any positive integer n, we consider

$$\int_0^{\pi/2} \cos^{2n} t\,dt.$$

Applying integration by parts twice, we obtain

$$\begin{aligned}\int_0^{\pi/2} \cos^{2n} t\,dt &= [t \cos^{2n} t]_0^{\pi/2} + 2n \int_0^{\pi/2} t \cos^{2n-1} t \sin t\,dt \\ &= n[t^2 \cos^{2n-1} t \sin t]_0^{\pi/2} - n \int_0^{\pi/2} t^2[-(2n-1)\cos^{2n-2} t \sin^2 t + \cos^{2n} t]\,dt \\ &= -2n^2 \int_0^{\pi/2} t^2 \cos^{2n} t\,dt + n(2n-1) \int_0^{\pi/2} t^2 \cos^{2n-2} t\,dt \\ &= -2n^2 I_{2n} + n(2n-1) I_{2n-2},\end{aligned}$$

where $I_{2n} = \int_0^{\pi/2} t^2 \cos^{2n} t\,dt$. Hence

$$-2n^2 I_{2n} + n(2n-1) I_{2n-2} = \int_0^{\pi/2} \cos^{2n} t\,dt.$$

As is known, (see, for example, [2], p. 226)

$$\int_0^{\pi/2} \cos^{2n} t\,dt = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2},$$

where, as usual,

$$(2n)!! = 2\cdot 4\cdots(2n-2)(2n), \quad 0!! = 1;$$
$$(2n+1)!! = 1\cdot 3\cdots(2n-1)(2n+1), \quad (-1)!! = 1.$$

Thus we have

$$-2n^2 I_{2n} + n(2n-1)I_{2n-2} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2},$$
$$\frac{(2n)!!}{(2n-1)!!}I_{2n} - \frac{(2n-2)!!}{(2n-3)!!}I_{2n-2} = -\frac{\pi}{4}\frac{1}{n^2}.$$

This implies that

$$\frac{(2n)!!}{(2n-1)!!}I_{2n} - \frac{0!!}{(-1)!!}I_0 = \sum_{k=1}^{n}\left[\frac{(2k)!!}{(2k-1)!!}I_{2k} - \frac{(2k-2)!!}{(2k-3)!!}I_{2k-2}\right]$$
$$= -\frac{\pi}{4}\sum_{k=1}^{n}\frac{1}{k^2},$$

and hence that

$$\frac{(2n)!!}{(2n-1)!!}I_{2n} = \frac{\pi^2}{24} - \frac{\pi}{4}\sum_{k=1}^{n}\frac{1}{k^2} = \frac{\pi}{4}\left[\frac{\pi^2}{6} - \sum_{k=1}^{n}\frac{1}{k^2}\right].$$

It is sufficient to prove that

$$\lim_{n\to\infty}\frac{(2n)!!}{(2n-1)!!}I_{2n} = 0. \tag{1}$$

Now we have

$$I_{2n} = \int_0^{\pi/2} t^2\cos^{2n}t\,dt \leqq \left(\frac{\pi}{2}\right)^2\int_0^{\pi/2}\sin^2 t\cos^{2n}t\,dt$$
$$= \frac{\pi^2}{4}\left[\int_0^{\pi/2}\cos^{2n}t\,dt - \int_0^{\pi/2}\cos^{2n+2}t\,dt\right]$$
$$= \frac{\pi^2}{8}\left[\frac{(2n-1)!!}{(2n)!!} - \frac{(2n+1)!!}{(2n+2)!!}\right] = \frac{\pi^2}{8}\frac{(2n-1)!!}{(2n+2)!!}.$$

Therefore

$$0 < \frac{(2n)!!}{(2n-1)!!}I_{2n} \leqq \frac{\pi^2}{8}\frac{1}{2n+2}.$$

Thus we establish (1); hence the formula is proved.

Finally, the author wishes to express his thanks to a referee for valuable criticism.

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CLASSROOM NOTES

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AN ABSTRACT FORMULATION OF A PROBLEM RELATED TO GOLDBACH'S CONJECTURE

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0. Introduction. Goldbach's conjecture [1] states that every even integer greater than four can be represented as the sum of two odd prime numbers. This paper gives an abstract formulation of a problem that is at least as general as Goldbach's conjecture. Specifically the problem deals with a minimal condition associated with subsets of odd integers in the additive semigroup of strictly positive integers.

1. Definitions.

DEFINITION 1.1. *Let $(A, *)$ denote a nonempty set A together with a closed binary composition law "$*$" defined on A. Call $(A, *)$ an algebraic system. By a mutant of $(A, *)$ is meant a subset M of A that satisfies the condition that $M * M \subseteq \overline{M}$, where $M * M = \{a * b : a \in M, b \in M\}$ and $\overline{M}$ is the set of all of the elements of A not in M. Clearly M must be a proper subset of A. If and only if all of the elements of A are idempotent with respect to "$*$" let the empty set be the only mutant of $(A, *)$. With this convention every algebraic system has a mutant.*

DEFINITION 1.2. *A mutant M of $(A, *)$ is said to be a maximal mutant of $(A, *)$ if there is no mutant of $(A, *)$ which properly contains M. If and only if all of the elements of A are idempotent with respect to "$*$" let the empty set be the only maximal mutant of $(A, *)$. With this convention the Hausdorff extremal principle [2] guarantees that every algebraic system has a maximal mutant.*

DEFINITION 1.3. *Let N be a maximal mutant of $(A, *)$. If there exists a nonempty class of mutants $M_i \subseteq N$ of $(A, *)$ such that $M_i * M_i \supseteq \overline{N}$, for all $i \in I$, then $\bigcap_{i \in I} M_i$ is said to be a potential minimax mutant in N of $(A, *)$. When N is potential minimax mutant in N of $(A, *)$ it is said to be the trivial potential minimax mutant in N of $(A, *)$.*

DEFINITION 1.4. *Let N be a maximal mutant with a potential minimax mutant in N of $(A, *)$. A minimax mutant in N of $(A, *)$ is a potential minimax mutant in N of $(A, *)$ for which there is no potential minimax mutant in N of $(A, *)$ that is properly contained in it. When N is a minimax mutant in N of $(A, *)$, it is the only minimax mutant in N of $(A, *)$. Then N is said to be the trivial minimax mutant in N of $(A, *)$.*

2. Propositions.

LEMMA. *No nontrivial potential minimax mutant in N of $(A, *)$ can be a maximal mutant of $(A, *)$.*

Proof. No proper subset of a mutant can be a maximal mutant.

THEOREM 2.1. *The additive semigroup $(A, +)$ of strictly positive integers has a nontrivial minimax mutant in N, where N is the set of all positive odd integers.*

Proof. Clearly N is a maximal mutant of $(A, +)$. Since $N+N=\overline{N}$ then $N+N\supseteq\overline{N}$. Put $N^*=N\cap\{\bar{n}\}$, where $3<n\in N$. It is easy to show that this proper subset of N satisfies the condition $N^*+N^*\supseteq\overline{N}$. Now apply the Hausdorff extremal principle [2] to establish the existence of a nontrivial minimax mutant in N of $(A, +)$. When applying the Hausdorff extremal principal observe that any finite set is a suitable lower bound for the required simple ordering (set inclusion) of potential minimax mutants in N of $(A, +)$.

THEOREM 2.2. *Not every maximal mutant of the additive semigroup $(A, +)$ of strictly positive integers has a potential minimax mutant in it.*

Proof. Clearly the set $S=\{2+4n: n=0, 1, 2, \cdots\}$ is a maximal mutant of $(A, +)$. But $S+S\subseteq E$, where E is the set of all positive even integers. Hence no odd integer can be represented as the sum of two elements of S.

Conclusions. In terms of the present terminology which involves general concepts with many other properties than the ones presented here [3], if Goldbach's conjecture is a valid theorem, it represents, among other things, another existence proof for a nontrivial potential minimax mutant in the set of all positive odd integers with respect to the additive semigroup of positive integers.

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ON THE APPROXIMATION OF IRRATIONALS

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This note contains an unexpected application of Lebesgue's theorem on the differentiability of a function of bounded variation to the theory of approximation of irrationals by rationals. For an exposition of Lebesgue's theorem the reader is referred to [3], Chapter 1. The Carus Monograph by Boas contains an excellent exposition of the difficult part of this theorem, *i.e.*, its proof for monotone functions ([1], p. 134).

THEOREM ([2], Theorem 198). *Let χ be a positive-valued function defined on the positive integers. Let E be the set of irrationals ξ such that $|\xi - p/q| < 1/q\chi(q)$ for infinitely many q and appropriate p. If the series $\sum_{q=1}^{\infty} 1/\chi(q)$ converges then E has Lebesgue measure zero.*

We have adopted the convention that an expression like p/q shall always denote a quotient of integers with $q>0$ and $(p, q)=1$.

Proof. Without loss of generality we restrict our attention to the interval $[0, 1]$ where we define f by $f(x)=0$ if x is irrational, $=1/q\chi(q)$ if $x=p/q$. Since there are fewer than q fractions with denominator q (except for $q=1$, when there are two), any sum of the form $\sum_{k=1}^{n} |f(x_k)-f(x_{k-1})|$ with $0=x_0<x_1< \cdots <x_n=1$ is easily bounded by $2\sum_{q=1}^{n} 1/\chi(q)$, so f is of bounded variation. Thus, by Lebesgue's theorem, f is differentiable almost everywhere and hence differentiable for almost all irrationals.

Let ξ be an irrational for which $f'(\xi)$ exists. Then by considering a sequence of irrationals converging to ξ we find $f'(\xi)=0$. Now consider rationals p/q, close to ξ. We have

$$\frac{f(p/q)-f(\xi)}{p/q-\xi} = \frac{1}{q\chi(q)(p/q-\xi)},$$

and this quantity is small when p/q is close to ξ. Thus, except for finitely many q, $q\chi(q)|p/q-\xi|>1$, and ξ cannot be in the set E. Thus E contains only points where f' does not exist and hence E has measure zero.

This is certainly not the simplest proof of this theorem (see [2]) but with appropriate hints the construction of this proof becomes a good exercise for students acquainted with Lebesgue's theorem.

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A NOTE ON INDETERMINATE FORMS

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In the teaching of indeterminate forms in sophomore calculus, it has been noted that in many textbooks, in the handling of the form 0^0, that examples of the type $\lim_{x\to 0} x^x$ and $\lim_{x\to 0} (\sin x)^{\tan x}$, which invariably tend to a limit of 1, are the only ones given. The good student is therefore inclined to ask, if such seems always the case, why is it necessary to say that $a^0 = 1$, provided that $a \neq 0$. This has led to the construction of the following simple example.

Consider

$$\lim_{x\to 0^+} x^{a/\ln x}, \ a \text{ some constant.}$$

Then, by the usual procedure with such forms, let

$$z = x^{a/\ln x}.$$

Thence, taking logarithms,

$$\ln z = \frac{a}{\ln x} \cdot \ln x = a,$$

and hence

$$\lim_{x\to 0^+} \ln z = a.$$

Making the allowable interchange of operations,

$$\lim_{x\to 0^+} z = \lim_{x\to 0^+} x^{a/\ln x} = e^a,$$

and the limit may be varied at will by the choice of a. This is usually satisfactory for the good student.

In an effort to generalize this relatively simple result, let us examine the following theorem.

THEOREM. *If $f(x)$ is continuous and of the order of x at the origin (i.e., $\lim_{x\to 0} (f(x)/x) = b$, $0 < b < \infty$), if $g(x)$ is continuous at the origin, and if $h(x)$ is of the order of $\ln x$ in a right-hand neighborhood of the origin ($\lim_{x\to 0^+} (h(x)/\ln x) = c$, $0 < c < \infty$), and if these functions have the necessary derivative properties for the use of L'Hospital's rule, then*

$$\lim [f(x)]^{g(x)/h(x)} = e^{g(0)/c}. \tag{1}$$

Proof. Since $f(x)/x$ is given to be an indeterminate form of 0/0 type, one application of L'Hospital's rule will yield

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{f'(x)}{1} = b. \tag{2}$$

Similarly, it is known that $h(x)/\ln x$ is an indeterminate form of ∞/∞ type, and hence

$$\lim_{x\to 0^+} \frac{h(x)}{\ln x} = \lim_{x\to 0^+} \frac{h'(x)}{1/x} = c;$$

that is,

$$\lim_{x\to 0^+} xh'(x) = c. \tag{3}$$

Now, following the usual procedure, let

$$z = [f(x)]^{g(x)/h(x)}.$$

Then

$$\ln z = \frac{g(x)}{h(x)} \ln f(x),$$

and

$$\begin{aligned} \lim_{x\to 0^+} \ln z &= \lim_{x\to 0^+} \frac{g(x)}{h(x)} \ln f(x) = \lim_{x\to 0^+} g(x) \cdot \lim_{x\to 0^+} \frac{\ln f(x)}{h(x)} \\ &= g(0) \cdot \lim_{x\to 0^+} \frac{f'(x)/f(x)}{h'(x)} = g(0) \cdot \lim_{x\to 0^+} \frac{f'(x)(x/f(x))}{xh'(x)}. \end{aligned} \tag{4}$$

If we now make use of (2) and (3), then (4) becomes

$$\lim_{x\to 0^+} \ln z = \frac{g(0)\cdot(b)\cdot(1/b)}{c} = \frac{g(0)}{c}.$$

Thence, making use of the allowable interchange of limit and logarithm, we get

$$\lim_{x\to 0^+} z = \lim_{x\to 0^+} [f(x)]^{g(x)/h(x)} = e^{g(0)/c},$$

and (1) is established.

As an example of this more general result, consider

$$\lim_{x\to 0^+} [\sin bx]^{aJ_0(x)/Y_0(x)},$$

where a and b are constants, $J_0(x)$ is the well-known first solution of the Bessel equation of order zero ($x^2y''+xy'+x^2y=0$), and $Y_0(x)$ is a second solution of this equation, which, for definiteness, may be taken in the Neumann form, *viz.*,

$$Y_0(x) = J_0(x) \ln x + \frac{x^2}{2^2} - \frac{x^4}{2^2\cdot 4^2}\left[1+\frac{1}{2}\right] + \cdots.$$

The functions $\sin bx$, $aJ_0(x)$, and $Y_0(x)$ have the properties ascribed to $f(x)$,

$g(x)$, and $h(x)$. Applying the theorem directly, we can show that

$$\lim_{x \to 0^+} [\sin bx]^{aJ_0(x)/Y_0(x)} = e^a.$$

Since a is arbitrary, we may produce any limit we please. It is interesting to note that the parameter b apparently does not affect the result. In this last example c is unity, and $J_0(0) = 1$.

(Note: This paper was presented before the Mathematics Section of the North Carolina Academy of Science, May 7, 1960.)

MATHEMATICAL EDUCATION NOTES

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THE ADVANCED PLACEMENT PROGRAM IN MATHEMATICS

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It is well known that there is an increasing demand from our society for scientists and other personnel who are competent in mathematics. This demand is not being completely met, but, finally, something significant is being done in the direction of meeting it. One of the many significant responses to this need is the College Entrance Examination Board Advanced Placement Program in mathematics.

This country has been faced for some time with the important responsibility of developing more effectively the potentialities of students who have capabilities and interests in mathematics. More specifically, our country's immediate problem is one of spotting these students, showing them what mathematics really is, helping them to develop interest in mathematics, and in one way or another giving them a program that will be effective in producing the competence desired.

Unfortunately, over the first half of the twentieth century, the content and organization of high school mathematics for college preparatory students in this country remained essentially unchanged—we refer to the traditional presentation of algebra, geometry, and trigonometry. It is true that certain modifications of teaching methods and some minor changes in topical emphasis occurred as a result of the 1923 Committee on Mathematical Requirements and the 1944 Commission on Post-War Plans of the National Council of Teachers of Mathematics. But nothing really significant took place.

However, the pressure had built up, and the recent developments and positive actions that have taken place in the 1952–60 period have not come as a surprise, however significant, important and helpful they may be. What has

taken place in this period was inevitable. But the first of all these programs to, in a sense, take the leadership and set the direction for the change in secondary school curriculum was the program which is now known as the Advanced Placement Program of the College Entrance Examination Board.

In 1952 grants by the Ford Foundation made possible three different studies of the problem of enriching and/or accelerating the education of gifted children in school and college. Of these studies the one most directly of concern to us now is the School and College Study of Admission with Advanced Standing which was under the Directorship of Dr. William Cornog, then Principal of the Central High School of Philadelphia, and more recently Superintendent of Schools at Winnetka, Illinois. Professor H. W. Brinkmann of Swarthmore College was chosen as the head of the mathematics subject matter committee. The group invited to participate in this project consisted of twelve different colleges* and was supplemented by a group of twelve representatives of schools, including teachers, principals, and superintendents. The group spent that academic year discussing the particular problem presented, that of designing a course at the secondary school level that would be the equivalent of one year of college work.

The problem presented was not a simple one. New frontiers in mathematics were creating almost unlimited opportunities for growth in mathematical knowledge and its applications to the physical sciences and engineering, to the social and biological sciences, and to business and industry. Instead of emphasis on the applications and the utility aspect of mathematics at this level, the nature of the subject, its generality and open-ended quality, its widely divergent fields, its system of logic and the fact of its continuous and lively growth needed to be emphasized. In order to accomplish this, the committee felt that the entire secondary school curriculum should be changed, and drew up a tentative syllabus† for such a three year program, terminating in "the Advanced Placement Course," one full year of college calculus with analytic geometry. Although this is not always the first year of college mathematics for the well-prepared student in the typical good college or university at the present time, a year of such mathematics does come into the college curriculum at some level. This will continue to be the case for many years to come.

When the first year of mathematics taught in the twelve original colleges of the School and College Study was considered, only four of these colleges covered precisely this material as a freshman course. Some also taught algebra and trigonometry. Others introduced considerable mathematical logic, notion of proof, and elementary set theory, before continuing with the calculus. The designed course, however, seemed to be the appropriate one, and appeared to be the most logical program to follow. The fact that such material does, in actual-

* Bowdoin, Brown, Carleton, Haverford, Kenyon, M.I.T., Middlebury, Oberlin, Swarthmore, Wabash, Wesleyan, and Williams.

† This syllabus appeared in the 1955 edition of The Advanced Placement Program: Course Description, College Entrance Examination Board. In the present 1960 edition, only the final year course syllabus is given.

ity, constitute one full year of college mathematics has made the problem of credit and placement a relatively easy matter. Some colleges have been able to place the advanced placement students in their regular first semester course in logic, from which the student would jump into the second semester sophomore course, and at the same time take advantage of the college's special curriculum, as well as the advanced placement course. In some cases, the students receive only one semester of placement or credit, while in others, students have covered a year and a half of a specific college course. One fact is clear. If the individual college desires to work with these students, the details of placing them at the appropriate place or level is quite possible. No other one year of college mathematics would in any sense make this placement possible. Thus, the Advanced Placement Syllabus has received almost universal acceptance and its present course description appears to be agreeable to all. The present mathematics syllabus, which appears in the *Advanced Placement Program: Course Description*,* follows:

The following is a check list of topics to be covered by the end of the year. It is not to be assumed that these topics are necessarily to be taken up in the order printed here but the list does indicate the scope of the Advanced Placement Examination in Mathematics.

1. Analytic geometry review and extension: rectangular and polar coordinates, distance and slope, parallelism and perpendicularity of lines, equations and graphs, line and circle, and other conics.
2. Differential calculus of algebraic functions: the function concept, absolute values and inequalities, definition and basic properties of limits, fundamental ideas of continuity, slope of a curve, average and instantaneous rates of change, definition of the derivative, formal differentiation, implicit functions and implicit differentiation, differentiation of composite functions and of parametrically defined functions, higher order derivatives, the differential and its use in approximation, and Rolle's theorem and the theorem of the mean.
3. Applications of differential calculus: tangents and normals, curve tracing, maximum and minimum values, both relative and absolute, rate problems and related rates, velocity and acceleration of a particle along a straight line and along a curve, and properties of conic sections involving tangents.
4. Integral calculus of algebraic functions: the inverse of differentiation, integration of simple expressions, basic formulas, integration by substitution, simple differential equations with initial conditions, intuitive development of the definite integral as the limit of a sum, intuitive treatment of the fundamental theorem of the integral calculus, evaluation of simple definite integrals, and approximation of definite integrals by the trapezoidal rule.
5. Geometric and physical applications of integration: the area under a curve, the average (mean) value of a function, areas between curves, volumes of revolution, volumes by slicing, motion in a straight line, and work.
6. The calculus of elementary transcendental functions: exponents and logarithms; the exponential and logarithmic functions; the inverse relationship of these functions; the derivatives of e^u, $\log_e u$ and $\log_a u$; the integrals of $e^u du$ and du/u; trigonometric functions of real numbers; limit of $(\sin x)/x$ as $x \to 0$; the derivatives and integrals of $\sin u$ and $\cos u$; the derivatives and integrals of other trigonometric functions; parametric representations involving trigonometric functions for curves such as the ellipse, hyperbola, and cycloid; the applications as listed above, when the functions involved are transcendental, including

* College Entrance Examination Board, 475 Riverside Dr., New York, 27.

growth and simple harmonic motion; and use of simple trigonometric substitutions in integration.

To prepare for such a program in the 12th grade, consolidation and revision of much material formerly taught in the secondary school curriculum is necessary, as well as, in some cases, acceleration.

A most interesting observation of many of the secondary school programs which have been designed might be made. All seem to follow the same general pattern with similar emphasis on the deductive method and logical approach to the material, and all agree that the Advanced Placement Program, as it is now outlined, has an important place for some students in the general organization of the entire curriculum.

The final report of the Commission of the College Entrance Examination Board has recommended it, and suggests that it either follow the completion of the regular Commission Program or be presented after the first half of the twelfth year in certain special cases. Also, the Illinois Project and the School Mathematics Study Group material fit directly into the Advanced Placement Program and have had students do satisfactory work on the examination.

At the time the original program, under the Fund for the Advancement of Education, was turned over to the College Board (1955), and became the present Advanced Placement Program, some of those in the mathematical world, aware of what had taken place and appreciative of the possibilities for the unbelievable growth of the Program in mathematics, wished to have their feelings known. Specifically, at the Evaluation Conference held at Williams College, June 22–25, 1955 the invited group unanimously adopted the following significant resolution:

"After conferring together and evaluating, where possible, the impact of the 1954 report of the Mathematics Subcommittee, we, the members of this conference, are unanimously agreed that, in view of the relatively small number of students who have thus far benefited from the program carried on under the report; in view of the long-range planning required by most schools before a change in curriculum and a full test of the program can be made; in view of the considerable number of schools here represented who wish to undertake to implement the subcommittee's report in the coming year; and, in view of the potential enrichment of the mathematical training at all levels which may result, it is apparent that many worthwhile gains would be lost if the 1954 report were now cast loose and left to make its own way at this critical time.

We therefore respectfully recommend that the present subcommittee be continued. We recommend this in order that this experienced group may:

(1) carry out a vigorous continuation of the program,
(2) conduct systematic evaluations of the program's results in the future,
(3) publicize the facts regarding the program among schools and colleges,
(4) work with the College Entrance Examination Board in ways which will help the program fulfill its objectives, and
(5) cooperate with other organizations and agencies who are interested in like problems."

Now, five years later, we recognize the importance of the general acceptance of this resolution, how it has been carried out, and discover the almost unbelievable results. The College Entrance Examination Board has created a committee of mathematicians and teachers of mathematics who have not only outlined

the program and made out the syllabus, but have taken the responsibility of writing an appropriate examination for this material each year. Although changes may take place in the years ahead, in recent years this examination has been divided into two parts. Part I consists of thirty multiple choice questions to be covered in one hour. Part II consists of ten longer problems for which the candidate was expected to write out detailed solutions of as many as possible in two hours. The first part is machine-marked, while the second is marked by a group of competent college and secondary school teachers selected by the Educational Testing Service specifically for this work. The test is given in May at the secondary schools and is marked in June. The student's Part II paper, his grade, an outline of the actual course taken in secondary school, and the teacher's specific recommendation are then sent to the college which the student plans to attend. On the basis of this information the college decides whether credit and/or placement is to be given the student.

Below are listed the results of the candidates participating since 1955, which indicate the growth of the program. From these results, especially those for 1960, it should be apparent that students should *not* be permitted to take the examination unless they are prepared in the course listed in the syllabus. Five is the highest grade, while one is the lowest.

Grade	Number of Candidates each year					
	1955	1956	1957	1958	1959	1960
5	5	44	80	115	225	176
4	14	109	169	235	283	404
3	30	87	176	334	586	606
2	32	100	117	217	329	711
1	25	46	182	276	447	1011
Totals	106	386	724	1177	1870	2908

In addition, each year a follow-up study on the placement of candidates is made. The following table shows what took place in the 1585 cases (out of 1870) which were reported in 1959.

Grade	Number of Candidates	Number Awarded Credit and/or Placement	Number Awarded Neither
5	196	188	8
4	254	236	18
3	510	371	139
2	276	70	206
1	349	22	327
Totals	1585	887	698

These figures alone speak strongly enough for the success of the Advanced Placement Program in Mathematics, and for the mandate to continue it, with as much publicity as possible.

The Advanced Placement Program Conferences have been an integral part of the success of the program in mathematics. These have been held annually in June at Williams College, Phillips-Exeter Academy, Oberlin College, Wesleyan University, Ripon College, and Case Institute of Technology. Since the increase in attendance and the enthusiasm of the participants have been very significant, similar conferences are planned for the future. This June the conference will be at the University of Kansas. Both college and secondary school personnel who are interested in learning more about the program are encouraged to attend.

Since the program has been outlined as a three-year program, there are even now many schools which have entered into the work of the syllabus, but have not completed the three-year cycle, so that they have not been counted as formally working with the program, since they have as yet had no students take the test. With the increase in general emphasis on a stronger mathematics curriculum and the introduction of algebra in the middle, or beginning of the eighth grade, the numbers of successful candidates will increase at an even more rapid rate in the near future. Many observers feel that, in fact, within ten or fifteen years, what is now the Advanced Placement Program in Mathematics will become the regular standard course for the well-prepared college-bound secondary school student. If, in fact, such proves to be the case, the contribution of this program in the field of mathematics education at the secondary school level would be considerably more than originally imagined.

In order for the reader to have as clear a picture of the program as possible, the College Entrance Examination Board and Educational Testing Service have permitted Part I of the 1958 test and Part II of the 1959 test to be published. These test questions will appear in this Department next month.

THE ALL PROJECT (ACCELERATED LEARNING OF LOGIC)

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In American education today there is a need for better methods of elementary-school instruction and a need for early development of the skills of deductive reasoning so important in learning mathematics. The ALL Project (Accelerated Learning of Logic), established at Yale University under a three-year grant from the Carnegie Corporation, is an attempt to help meet both of these needs by (1) adapting and developing further certain new methods of instruction and (2) introducing mathematical logic into the elementary-school curriculum. In the studies undertaken in the ALL Project there will be investigated (1) methods based on those developed by Professor B. F. Skinner, Harvard University (See B. F. Skinner, *Cumulative Record*, 1959, pp. 143–182), and (2) a series of games, which is currently being developed by the ALL Project research staff.

The primary objective of the ALL Project is to develop and test materials and methods for teaching mathematical logic to elementary-school children. In pursuing this objective, information about the following will be sought:

(1) The characteristics of optimum Skinner programs for specified age and ability groups.
(2) The lowest age level for which the developed techniques for teaching mathematical logic can profitably be used.
(3) The optimum way of using the developed techniques, specifying (a) the equipment to be used, (b) the appropriate combination of Skinner-program work, game playing, and other modes of instruction, and (c) the appropriate amount of time to be devoted to the subject-matter.
(4) The costs of teacher-training, equipment, and materials for the techniques developed.
(5) The impact of this instruction on students in terms of their performance in other subject-matters, particularly mathematics.

It is anticipated that the methods developed can be adapted to subject-matters other than logic, but this adaptation is not an immediate objective of the ALL Project.

The initial studies of the ALL Project will be divided into four stages:

(1) The initial preparation of materials and determination of methods for administering them.
(2) The initial trial of prepared materials and modification of materials and methods in the light of acquired experience.
(3) The trial of modified materials with a larger and more varied sample group of students.
(4) The final modification of materials and report of findings, evaluations, and recommendations.

A "Skinner learning program" (in the terminology used here) presents subject-matter which requires the learner to make written responses to the material as it is presented. This method of presentation insures the learner's active participation. Answers are presented immediately after a response is made; there may be a representation of those parts of the material where incorrect responses occur. The Skinner learning programs are intended to be self-contained and self-explanatory, so that the learner while working through a program requires neither supplementary texts nor guidance from a teacher. This method of presentation allows each learner to proceed at his own pace. Since there is immediate reinforcement of correct responses, and immediate correction of inappropriate responses, a Skinner learning program affords one of the chief advantages of a private tutor. A record of the learner's errors shows the teacher or programmer where the presentation is weak or obscure and where each learner has particular difficulty. Skinner learning programs for mathematical logic will be especially useful, because they can be used in training teachers as well as in teaching students; such training of teachers will probably be necessary before this novel subject-matter can be introduced into the elementary school curriculum.

The series of games that is being developed has as its goal the teaching—in a competitive and entertaining atmosphere—some of the symbol-manipulating skills of logic. The games are graduated in their complexity. The player is taught first how to form sentences in a special "language," and then how to show which

of these sentences are "true," where the "language" is a system of formal logic and the "true sentences" are theorems. The following factors seem likely to make the games especially suitable for facilitating learning:

(1) Practice in logic comes as a by-product of an activity which is enjoyable in itself.
(2) Although young children enjoy these games, the more intricate ones pose a genuine challenge even for adults.
(3) In the play of the games there is no waiting time. Each player proceeds at his own pace throughout the entire time of play. The more adept player is not delayed by those who play more slowly.
(4) Everyone else in these games learns from the best player. His strategies are displayed openly so that all others may learn to adopt them. In effect, the best players act as teachers, although they are not formally assigned this role by the rules of the games.
(5) The games are so ordered that each new game is slightly more intricate than the previous one and each later game uses the skills learned in earlier games.
(6) The games emphasize individual rather than collective decision making. Each player plans and executes his own strategy independently in order to achieve specified goals.
(7) The games are flexible both in the number of persons who may play (two or more) and in the length of time for a game to be played (five minutes or more).

The number of such games and the range of the subject-matter that can be taught by them is still an open question. It does seem clear, however, that even for mathematical logic, the number of such games will be quite large. So far, only the first twelve games have been developed to the stage of constructing prototypes of them. The initial games in the series have already been developed and tried informally with groups of children and adults, including several faculty members and graduate students of Yale University; their responses have been enthusiastic.

When the materials have been initially developed, they will be tested by administering them to a small sample of sixth graders. Teaching machines and programmed textbooks will be used for displaying the prepared materials; this presentation will be supplemented with the proposed games and perhaps, some other traditional instruction. This initial trial will help to determine:

(1) The appropriate machine-to-student ratio.
(2) In what proportion machine instruction, games, and traditional instruction should be combined.
(3) The approximate time to be spent on the subject-matter (how long sessions should be, how frequently they should occur, and how many sessions should be given).
(4) How materials need to be modified (a) for effective use in sixth grade and (b) for use with learners of different ages and abilities.

After the materials are modified in light of the initial trial, they will be given to a larger sample of students, including children younger than those in sixth grade.

The last phase of this initial stage will involve a final modification of materials and methods. A report will be prepared containing information on

(1) Administration procedures, teacher-training, and procurement and preparation of equipment and material.

(2) Estimated outlay in money and in time of both students and teachers, and
(3) The effect of this training in logic on the students' progress in acquiring other intellectual skills.

The plan is to work in schools where achievement tests are administered annually. Comparisons of the year-to-year improvement in achievement test scores in various subject-matters—*e.g.*, mathematics, language skills (grammar), and reading comprehension—will be made between students who are given this training in logic and those who are not. These comparisons should furnish valuable information about the desirability of arranging a large-scale controlled experiment for testing the effects of introducing mathematical logic in the elementary-school curriculum by means of the materials and methods developed in this study.

The principal investigator for the ALL Project is Layman E. Allen, Assistant Professor of Law, Yale University. Frederic B. Fitch, Professor of Philosophy, Yale University, will serve as senior consultant. The research staff includes Robin B. S. Brooks, James W. Dickoff, and Patricia A. James.

N.S.F. SUMMER INSTITUTES IN MATHEMATICS: THE VISITING FOREIGN STAFF PROGRAM

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In 1958 (and even earlier), a handful of eminent foreign mathematicians were included in the staffs of the relatively few summer institutes for high school mathematics and science teachers. Based on the experiences gained in this way, a broader and possibly more purposeful Visiting Foreign Staff Program for summer institutes was conceived and supported in major part by the National Science Foundation. This program had three parts, *viz.*, 1) Biological Sciences, 2) Mathematics, and 3) Physical Sciences. In 1959 these programs were operated independently and with little mutual cooperation by three different colleges and universities. In 1960 there was a project coordinator with a resulting increase in the cooperation of the three parts again operated by three separate institutions. In 1961 the program is being operated by the American Association for the Advancement of Science.

As originally conceived, the mathematics part of the VFSP had as its central purposes the following: a) To acquaint eminent foreign mathematicians and mathematics educators with problems in mathematics education in the United States in the hope that constructive suggestions might be forthcoming; b) To explain to the VFSP personnel the several distinct (but in most cases correlated) new programs intended for curricula improvement (such as those of SMSG, UICSM, and so on) being formulated and tried in this country; c) To give the VFSP personnel some understanding of the overall summer institutes program, by description, observation, and participation, with the expectation that they would make significant and perhaps useful evaluative and constructive criticisms; d) To provide institute personnel, including participants, instructors, and

directors, with information concerning the situations and conditions in mathematics education in foreign countries; and e) To acquaint some of our teachers (and instructors) with the types and extents of preparation and other qualifications demanded of secondary school teachers of mathematics in other parts of the world.

Each visit for the foreign visiting lecturers in mathematics lasted for one week (longer in a very few cases). Each institute director was free to make whatever use he wished of the visitor's time. The most usual practices were i) to invite institute participants to seek informal contacts with the visitor, ii) to ask for one or more formal lectures on mathematics education in his country, iii) to ask for one or more lectures on mathematics, and iv) to leave much (and in some cases, all) of his time free for informal activities.

At the end of the summer, 1960, the VFSP personnel were reassembled, with consultants and resource personnel to compare experiences and write a formal report. In 1959 a report was written by the mathematics section alone and later distributed to all 1960 institute directors. In 1960, a joint report was issued by all three sections. The terminal activity was attendance at the August Conference of the National Council of Teachers of Mathematics at the University of Utah.

Although it is perhaps too early to evaluate the program, the following comments seem appropriate: a) In 1960, the program was more effective in all respects than in 1959; b) The reports issued were pertinent and contained some useful material; c) activity similar to the institute program has been carried out successfully in at least two foreign countries, and is contemplated in at least one other.

Report of Progress in Development of the Statewide Study of Instruction in Mathematics in California

1. **Delimitation of roles and responsibility.** The understanding of improvement of the quality of instruction in mathematics in the elementary and high schools throughout California required at the outset a careful determination of the respective responsibilities of the State Department of Education, of the administrators and teachers of school districts, and of the departments of mathematics in colleges and universities of the state.

The responsibility of the Department of Education through its Bureaus of Elementary and Secondary Education is (1) to assemble the best available information regarding what mathematics needs to be taught in the public schools, (2) to bring this body of content to the attentive consideration of mathematics teachers, (3) to encourage school districts to introduce and to experiment with revised subject-matter, (4) to enlist the active assistance of college mathematicians in providing such inservice re-education of teachers as may prove needful, and (5) finally, to coordinate the course offerings among and within districts so that pupil progress from grade to grade may be orderly and without impediment.

It is the responsibility of superintendents of school districts, of directors of curriculum, and of school principals (1) to free time for teachers from their classrooms so that they may thoroughly acquaint themselves with the recommended new materials, (2) to determine with their teachers what topics may immediately be incorporated into ongoing courses, (3) to postpone the introduction of unfamiliar concepts until the teachers

may have opportunity to master these, and (4) to provide means of in-service education so teachers may achieve this mastery.

The assistance of departments of mathematics in colleges and universities must be enlisted to enable teachers in public school service to gain insight into and familiarity with whatever processes and emphasis which a reformation of mathematical offerings in the elementary and high school grades necessitates. A development of a more appropriate sequence of undergraduate courses for a teaching major in school mathematics also deserves renewed attention.

Although the teaching of arithmetic in the elementary school grades urgently needs careful review, and the strengthening of the preparation of elementary school teachers in this subject is of primary importance, it is at the junior and senior high school levels that major changes of topic, sequence, and emphasis are most readily evident. Accordingly, the Bureau of Secondary Education has undertaken four steps to effect progress toward the achievement of improved offerings in secondary school mathematics.

2. **Achievement of a consensus among mathematicians as to recommended course content.** The first task undertaken by the Bureau was made necessary by the fact that until 1960, when the Conference Board of the Mathematical Sciences was formed, mathematicians have had no single organized voice to speak for the profession similar to the American Institute of Biological Sciences, for example, whose numerous member and affiliated societies have combined for a Biological Sciences Curriculum Study. Many competent high school teachers have been seriously concerned to know whether they should follow the recommendations of the School Mathematics Study Group, or the University of Illinois Committee on School Mathematics, or the Report of the Commission on Mathematics of the College Entrance Examination Board, or perhaps another group. In December 1959, therefore, the Bureau asked representatives from several of the principal centers to meet in California and to establish, if possible, a consensus of agreements on common elements for a sequence of courses.

William H. Meyer, University of Chicago, served as moderator of the sessions, and a statement was prepared with the cooperation of E. G. Begle, Yale University, Director of SMSG; Howard F. Fehr, Teachers College, Columbia University, for the CEEB report; John L. Kelley, University of California, Berkeley; R. L. Wilder, University of Michigan; and professors of mathematics from Sacramento State College, Stanford University, and the University of California, Los Angeles, and Santa Barbara. Following the meeting, Dr. Meyer circulated the statement among his colleagues for their criticisms and corrections. A final draft of recommendations appeared in the September, 1960, issue of *California Schools*, published by the Department of Education. Already it has stimulated review and study on the part of many school districts.

3. **Establishment of connection with the State Curriculum Commission study.** As a next step the Bureau of Secondary Education called a second meeting in May 1960, at which certain mathematicians and representatives of selected school districts were asked to advise the State Curriculum Commission as to procedures for an expanded study under its auspices. In order that members of the Curriculum Commission and the delegates from school districts might acquaint themselves with the findings of major studies of school mathematics, their recommendations were summarized in informal presentations by consultants present.

The further agenda of the May meeting invited the support of school districts "to undertake (1) consideration of the summaries of recommendations and instructional materials, (2) determination of what topics present in current mathematics courses should be lessened in emphasis or eliminated, and (3) arrangements for introduction of new focus on concepts and principles as rapidly as may be effected with reference to teacher confidence and security." Likewise the representatives of departments of mathe-

matics in the Universities and State Colleges of California were queried to ascertain what contributions their several institutions could make toward inservice education to promote the prompt utilization by elementary and high school teachers of the recommendations and new materials.

4. **June 20–30, 1960, workshop for teachers of secondary school mathematics.** A two-weeks' workshop was organized for late June to which districts were invited to send teachers. Eighty-seven teachers were designated by their superintendents to attend. At this workshop the participants were sectioned by secondary school grade levels: one section for teachers in grades 7–8, a second for grades 9–10, a third for grades 11–12, and a fourth which concerned itself with problems of the general-terminal student. Each section had two co-directors selected from districts already in the forefront of making advances in the reorganization of their mathematics courses.

Dr. Meyer was recalled to California to direct the actual workshop and from time to time other mathematical authorities spent several days with each section. These included Paul H. Daus, University of California, Los Angeles; Richard A. Dean, California Institute of Technology; and Professor Moredock, Sacramento State College.

Many of the participants at the June 20–30 workshop have organized study groups within their districts. Several have begun to plan regional week-end meetings for high school teachers in their areas. Probable locations of forthcoming meetings, in whose planning the Bureau has been asked to assist, are Santa Clara County, Shasta County, Sonoma County, Kern and Tulare counties, Stanislaus and Merced counties, Riverside and San Bernardino counties, and San Diego County.

(Abstracted from a report to members of the California State Curriculum Commission by Frank B. Lindsay.)

Resolutions by the Council of Chief State School Officers

At the annual meeting of the Council of Chief State School Officers, held in Santa Fe, New Mexico, November 18, 1960, a number of resolutions on education were passed. Those which seem of most direct interest and concern to mathematics education are the following:

Quality education. The Council believes that meeting the needs for more teachers and more school facilities, critical though this be in many parts of our nation, is not enough. The challenge of quality is even more significant than that of quantity. The Council urgently calls upon state and local fiscal authorities, legislative bodies, state departments of education, boards of education, and the supporting public to direct more energy and money into educational research and into the continued development of supervisory and administrative services at state and local levels.

Cost of higher education. The Council recognizes that there is greatly increased need for well-trained, well-informed and well-educated citizens. There is a shortage of trained manpower in almost every field of endeavor, making it imperative that provisions be made so that all competent high school graduates desiring to do so may have the advantages of college or university education.

The Council believes that tuition, student fees, and other costs to students in public institutions should be kept as minimal as possible so that no capable high school graduate will be denied the opportunity of a college education for economic reasons.

The Council also believes that the financial support of higher education should be greatly increased.

Amendment and extension of National Defense Education Act. The Council favors the appropriate amendment and extension of the National Defense Education Act, inas-

much as its operation as an emergency measure has been of considerable assistance to education and can continue to be of even more assistance in the future. The Council favors the following specific changes in the Act, and then its extension for a term of not less than four years. [The changes recommended for Titles III, IV, and VII only are included in this report.]

Title III—Materials and Equipment for Science, Mathematics and Modern Foreign Languages.

Amend Section 303 (a) (1) to include English, history, geography, economics, and government, along with the original fields of science, mathematics and modern foreign languages, as subjects of instruction in elementary and secondary schools that may be aided under this title.

Amend Section 305 to permit any funds for loans to private non-profit schools and unused by them to be made available for reallocation to the states for allotment under Section 302 (a) and payment under Section 304 (a).

In view of the expanded scope of the title, the Council favors an authorization of not less than $85 million annually under Section 301 for making payments to state educational agencies to carry out the programs described in paragraph (5) of Section 303 (a).

Title IV—Fellowships.

Provide a new section to authorize new types of fellowships, traineeships and leadership institutes for the preparation and in-service training of educational administrators in colleges and universities, state departments of education, and local school districts.

It is proposed that the appropriations for Title IV be increased approximately $40 million annually above the present authorization.

Title VII—Research and Experimentation in Mass Media.

Amend to include authorization for experimentation and research projects by state departments of education covering all or any part of the curriculum of the elementary and secondary schools, with authorization for appropriations of $15 million annually.

Shell Merit Fellowships

Introduction. The Shell Merit Fellowships for outstanding high school teachers of mathematics and science were established by the Shell Companies Foundation, Incorporated, as a part of their aid to education program. From the onset of the program in 1955, it was assumed that a greater impact would be made on the improvement of mathematics teaching in secondary schools if good teachers, well-trained academically (with a minimum of five years teaching experience) were trained to assume professional leadership at a local level. With this in mind, a special effort has been made to distribute the Fellowships to meritorious teachers systematically in each of the states and each of the provinces of Canada. Two universities, Stanford and Cornell, were selected to administer the Fellowships and to devise appropriate seminars to implement the policy.

Stanford University. At Stanford University the program for the mathematics teachers has been developed cooperatively with the Department of Mathematics and two of the seminars are taught there. The program over the years has varied somewhat but typically the work has consisted of courses on *Mathematical Methods in Science* or *Introduction to Modern Algebra* and a combined course on *Statistics and Probability*. These courses are intensive, meet five days per week, and continue for two months. The intent is to extend the preparation of the teachers rather than to review previously taken courses.

A seminar on *Recent Developments in Secondary School Mathematics* is given in the School of Education. The "new" programs in mathematics are examined and ways of bringing about curriculum changes in high schools are explored.

Four days a week a special seminar two to three hours in length is conducted by a research scientist or mathematician on new developments in his field of interest. George Pólya, Linus Pauling, J. van Overbeek, Emilio Segré, Gerald L. Pearson, Konrad Krauskopf, George Forsythe, Samuel Karlin, Wilson J. Frank, Robert F. Mozley, and Paul H. Kirkpatrick were among the seminar leaders in the 1960 program. The purpose of these seminars is to develop within teachers something of the mood and feeling for modern scientific intellectual activity. It was our assumption that this is best done by meeting and talking with men active in research.

The teachers live on the campus, eat and study together. The opportunity to discuss problems of mathematics teaching with other mathematics teachers has proven to be a significant aspect of the Shell Merit Seminars.

Cornell University. The special program at Cornell has followed a special study guide developed by Professor Emeritus Walter B. Carver. This guide has been revised for the last two summers of the Program. The work seeks to present new insights in algebra, geometry, trigonometry, limits and sequences, linear equations, problem solving, exponents and logarithms, calculus and modern mathematics. The course has met for double periods each day for six weeks for a total of 60 class periods. Opportunities to prepare instructional aids have also been available.

The mathematics teachers have been involved in one or more special seminars on teaching mathematics and have studied mathematics films and references. They have met with teachers of chemistry and physics for some of the broader instructional considerations. In the last five years renowned mathematicians and mathematics educators have had a part in the Program. Among these have been: Isadore Blumens, John A. Brown, Kenneth E. Brown, Harold Fawcett, Howard F. Fehr, P. R. Halmos, Phillip S. Jones, John Mayor, Paul C. Rosenbloom and Robert Walker. Several supervisors of mathematics have been among the teachers.

The mathematics teachers have at times been a part of instructional activities related to physics and all of them have become acquainted with the facilities, the course work and some of the research activities that make up the study of engineering. This has been done to help mathematics teachers better advise their students. All the mathematics teachers have had instruction relating to computers. They have developed the programs to solve certain problems and have had the opportunity to use a computer with the help of a technician. The understanding of computers has been further strengthened by a visit to the main plant of the International Business Machines Corporation. An understanding of the way mathematics is used in research has been heightened by a visit to the research centers of The Corning Glass Works and the Ansco Company.

The teachers live in a university dormitory where special library facilities are available. They eat their noon lunches and enjoy some special dinners and other social events with the entire group. These activities provide many opportunities to share ideas informally but there is also a special evening seminar each week when the mathematics teachers share ideas in an organized manner.

(The statements were prepared by W. M. Upchurch, Jr., Shell Companies Foundation; Paul DeH. Hurd, Stanford University; and Philip S. Johnson, Cornell University.)

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1466. *Proposed by H. Y. Shee, Huwei Middle School, Huwei, Taiwan, China*

Let $A_1 \cdots A_n$ be a cyclic polygon. Let B_i be the foot of the perpendicular from A_i on $A_{i-1}A_{i+1}$, C_i the point of intersection of A_iA_{i+2} and $A_{i-1}A_{i+1}$, and D_i the foot of the perpendicular from C_i on A_iA_{i+1}, where it is understood that all subscripts are to be reduced to the least positive residue modulo n. Show that

$$\prod_{i=1}^{n} A_{i-1}B_i/B_iA_{i+1} = \prod_{i=1}^{n} A_iD_i/D_iA_{i+1} = 1.$$

E 1467. *Proposed by D. G. de Figueiredo, New York University*

Write down the remainder of the division of $(x+1)^n$ by $(x-1)^3$.

E 1468. *Proposed by B. H. Bissinger, Lebanon Valley College*

Let $b_1, \cdots, b_n$ be any rearrangement of the positive numbers $a_1, \cdots, a_n$. Then $a_1/b_1 + \cdots + a_n/b_n \geqq n$.

E 1469. *Proposed by P. G. Kirmser, Kansas State University*

Find a function $f(x)$ such that $f(2x) = e^{x^2} f(x) \cos x$ with $f(0) = 1$.

E 1470. *Proposed by W. E. Patten, South Boston, Virginia*

It is desired to form a $2 \times n$ rectangle from 1×2 rectangles (dominoes), or we may say, to cover the rectangle with dominoes. In how many distinct ways can this be done, where two solutions are distinct when they cannot be brought into coincidence by rotations and reflections?

SOLUTIONS

A Triangle Formed by Three Cevians

E 1436 [1960, 922]. *Proposed by Mok-Kong Shen, Karlsruhe, W. Germany*

Through the vertices of a given $\triangle ABC$ draw straight lines l, m, n, respectively, such that n and l intersect in D, l and m in E, m and n in F inside $\triangle ABC$ and

(1) $\triangle ABE = \triangle BCF = \triangle CAD = \triangle DEF = (1/4)\triangle ABC$,

(2) $\triangle DEF$ is similar to $\triangle ABC$ and has an area equal to a given fraction of $\triangle ABC$.

An amalgam of solutions by Michael Goldberg, Washington, D. C., L. D. Goldstone, Watervliet, N. Y., and Beckham Martin, Toledo, Ohio. (1) Let l, m, n cut BC, CA, AB in U, V, W. We shall find a solution where $BU/UC=CV/VA=AW/WB$. Since a solution is affine, we first take triangle ABC equilateral. Draw lines parallel to the sides one-fourth the way from the sides to the opposite vertices. Through each pair of vertices draw internal arcs of radii equal to the circumradius. The intersections of the arcs with the lines give the points D, E, F. Note that two different triangles DEF can be so obtained. There result two possible positions of U on BC, and consequently two possible values of the ratio $k=BU/UC$. In the general case, where ABC is any triangle, we merely find the points U, V, W on BC, CA, AB such that $BU/UC=CV/VA=AW/WB=k$.

(2) Suppose, in similar triangles ABC and DEF, that $\measuredangle A=\measuredangle D$, $\measuredangle B=\measuredangle E$, $\measuredangle C=\measuredangle F$, and that $\triangle ABC/\triangle DEF=s^2$. We may easily draw circles ADC and AEB. Let AM and AN be diameters of these circles. Then

$$\begin{aligned} DE &= AE - AD = AN\cos(AN, AE) - AM\cos(AM, AE) \\ &= MN\cos(MN, AE) = AB/s. \end{aligned}$$

It follows tht $\cos(MN, AE)=AB/(sMN)$. Since MN is fixed, line ADE can be drawn, and D, E, F located. By differently pairing the equal angles in the two similar triangles ABC, DEF, six solutions can be obtained.

Also solved by Leon Bankoff, J. W. Clawson, J. M. Elkin, Jane Evans, S. H. Greene, D. C. B. Marsh, and S. W. Saunders. Late solutions by Dennis Couzin, T. R. Curry, and Guy Torchinelli.

Editorial Note. An easy computation shows that, in (1), $k=(3\pm\sqrt{5})/2$. The problem in (2) is that of placing a given triangle so that each side will contain one of three given points; this is Prob. 383 in Petersen's *Methods and Theories for the Solution of Problems of Geometrical Construction.*

A Convergent and a Divergent Series

E 1437 [1960, 922]. *Proposed by Nickolas Konopliv, University of Minnesota*

Test each of the following infinite series for convergence:

(1) $$x + \sin x + \sin(\sin x) + \sin[\sin(\sin x)] + \cdots,$$

(2) $$x - \sin x + \sin(\sin x) - \sin[\sin(\sin x)] + \cdots,$$

where $0<x<\pi$.

Solution by D. C. B. Marsh, Colorado School of Mines. (1) Denote the $(n+1)$st term by $\sin_n x$, $\sin_0 x$ being x. For $0<x<\pi$, $\sin_2 x=r$ lies (properly) between 0 and 1. Furthermore

$$\sin_3 x = \sin r > r - r^3/6 = r/2 + r(3 - r^2)/6 > r/2.$$

We can show inductively that $\sin_{n+1} x>r/n$ for $n\geqq 3$; for, from the hypothesis that $\sin_n x>r/(n-1)$, we find

$$\sin_{n+1} x = \sin(\sin_n x) > \sin[r/(n-1)]$$

$$> r/(n-1) - r^3/6(n-1)^3$$
$$= r/n + [r/n(n-1)][1 - nr^2/6(n-1)^2]$$
$$> r/n + [r/n(n-1)](1 - r^2) > r/n.$$

Thus the series dominates r times the harmonic series, and therefore diverges for all x, $0<x<\pi$.

(2) For $0<x<\pi$, the sequence $\{\sin_n x\}$ is monotone decreasing and bounded below. There is thus a limit L, with $L=\sin L$, implying $L=0$. By the alternating series test, (2) is seen to be convergent for all x, $0<x<\pi$.

Also solved by R. D. Adams, S. H. Greene, E. E. Lattman, C. S. Ogilvy, W. H. Ruckle, Norman Schaumberger, O. E. Stanaitis, C. E. Stenard, W. C. Waterhouse, and the proposer. Late solutions by P. R. Chernoff, Dennis Couzin, J. B. Muskat, J. L. Pietenpol, C. F. Pinzka, and I. D. Ruggles.

A Car Travelling Behind Another Car

E 1438 [1960, 922]. *Proposed by S. H. Gould, Mathematical Reviews, Providence, R. I.*

A car travelling at a speed of v miles per hour is required by law to remain at a distance (in Rhode Island one car-length for every 10 miles per hour) of cv miles behind the car ahead, where c is a constant depending upon the state. Find, as a function of time, the maximum allowable acceleration for a car starting from rest immediately behind an unobstructed car which accelerates at a constant rate of a miles per hour per hour.

Solution by R. W. Means, University of Santa Clara. The distance the second car travels as a function of time is described by the equation

$$x = (at^2)/2 - c(dx/dt).$$

Differentiating and letting $d^2x/dt^2=b$, which is the acceleration of the second car, one finds $b=a-c(db/dt)$. Solving this differential equation, with the given initial conditions, one obtains $b=a(1-e^{-t/c})$.

Also solved by L. R. Bragg, R. B. Brian, Gus Di Antonio, Michael Goldberg, S. H. Greene, Cornelius Groenewoud, Emil Grosswald, F. W. Herlihy, A. R. Hyde, Theodore Katsanis, M. E. Lakser, E. E. Lattman, L. A. MacColl, D. C. B. Marsh, M. V. Mielke, Norman Mines, D. A. Moran, Thomas O'Brien, David Rothman, Allen Rubenstein, David Sachs, Norman Schaumberger, R. T. Shannon, E. A. Sturley, Julius Vogel, W. C. Waterhouse, and R. H. Wilson, Jr. Late solutions by E. W. Brown, C. W. Dodge, J. L. Pietenpol, C. F. Pinzka, and E. L. Spitznagel, Jr.

For a Collection of Examples and Counterexamples

E 1439 [1960, 922]. *Proposed by J. M. Elkin, Long Island University*

Construct a bounded strictly monotonic function $f(x)$ such that $f'(x)$ exists for all real x and $\lim_{x\to\pm\infty} f'(x)\neq 0$.

Solution by D. A. Moran, University of Illinois. Let $f(n)=1-2^{-n}$ for nonnegative integers n. Extend the definition of f to all nonnegative real numbers to agree with the functions f_n defined on their respective domains as follows:

$f_n(x)$ is any monotonic differentiable function defined on $[n, n+1]$ which satisfies:

$$f_n(n) = f(n), \qquad f_n(n+1) = f(n+1),$$
$$f_n(n+1/2) = \{f(n) + f(n+1)\}/2,$$
$$f_n'(n) = f_n'(n+1) = 0, \qquad f_n'(n+1/2) = 1.$$

For example, certain arcs of two properly selected ellipses satisfy the conditions for the graphs of these functions in their respective intervals of definition.

Next, set $f(-x) = -f(x)$. Then f is defined for all real numbers, is bounded, is clearly strictly monotone, and $\lim_{x \to \pm\infty} f'(x) \neq 0$, since this limit does not exist, the derivative attaining the values 0 and 1 in each closed unit interval.

Also solved by M. T. Bird, J. L. Brown, Jr., P. M. Cohn, C. H. Cunkle, Bernard Fusaro, V. E. Hoggatt, Jr., D. C. B. Marsh, Hugh Noland, P. L. Renz, L. A. Ringenberg, David Rothman, W. H. Ruckle, Julius Vogel, W. C. Waterhouse, and the proposer. Late solutions by P. R. Chernoff, B. H. Dadbeh, J. B. Muskat, J. L. Pietenpol, E. L. Spitznagel, Jr., and D. C. Stevens.

Examples of undulating functions similar to the kind described above are

$$f(x) = 2^{(2^{n+1}-n-2)}(x-n)^{2^{n+1}} - 1 - 2^{-n}, \qquad n \leqq x \leqq n+1/2, \qquad n = 0, 1, 2, \cdots,$$
$$f(x) = -2^{(2^{n+1}-n-2)}(n+1-x)^{2^{n+1}} + 1 - 2^{-n-1}, \qquad n+1/2 \leqq x \leqq n+1, \qquad n = 0, 1, 2, \cdots,$$
$$f(x) = -f(-x), \qquad x < 0,$$

given by D. C. B. Marsh, and

$$f(0) = 0,$$
$$f(x) = f(n\pi) + 2^{-n}[1 - \cos^{1/(2n+1)}(x - n\pi)], \qquad n\pi < x \leqq (n+1)\pi, \qquad n = 0, 1, 2, \cdots,$$
$$f(x) = -f(-x), \qquad x < 0,$$

given by Bernard Fusaro.

A Closed-form Expression

E 1440 [1960, 922]. *Proposed by W. B. Carver, Cornell University*

Find a closed-form expression for $\sum_{s=1}^{n} s^3 r^{s-1}$, $r \neq 1$.

Solution by D. C. B. Marsh, Colorado School of Mines. Consider the more general expression, $f(m, n, r) = \sum_{s=1}^{n} s^m r^{s-1}$, with $r \neq 1$ and m a nonnegative integer. It is readily verified that

$$f(m, n, r) = (\partial/\partial r)[rf(m-1, n, r)], \qquad f(0, n, r) = (r^n - 1)/(r-1),$$

so that $f(m, n, r) = (Dr)^m[(r^n-1)/(r-1)]$, where $D = \partial/\partial r$. In the proposed problem $m = 3$, and the explicit expression is

$$\frac{n^3r^{n+3} - (3n^3+3n^2-3n+1)r^{n+2} + (3n^3+6n^2-4)r^{n+1} - (n+1)^3r^n + r^3 + 4r + 1}{(r-1)^4}.$$

Also solved by R. L. Bohuslou, L. R. Bragg, D. A. Breault, J. L. Brown, Jr., Gus Di Antonio, Ragnar Dybvik, J. M. Elkin, N. H. Fisher, Jr., W. W. Funkenbusch, Michael Goldberg, L. D. Goldstone, S. H. Greene, Cornelius Groenewoud, Emil Grosswald, Frank Herlihy, J. C. Hickman, J. E. Homer, Jr., A. R. Hyde, M. V. Mielke, D. A. Moran, C. S. Ogilvy, F. D. Parker, Thomas

Porsching, B. E. Rhoades, David Rothman, W. H. Ruckle, R. B. Sher and D. J. Uherka (jointly), F. C. Smith, M. R. Soliman, D. R. Sondergeld, O. E. Stanaitis, E. A. Sturley, Patrick Twomey, W. C. Waterhouse, Walter Zayachkowski, David Zeitlin, and the proposer. Late solutions by E. W. Brown, P. R. Chernoff, C. W. Dodge, J. B. Muskat, J. L. Pietenpol, and C. F. Pinzka.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4965. *Proposed by H. S. Shapiro, New York University*

Let $P(t_1, \cdots, t_n)$ be a homogeneous polynomial with real coefficients and D the partial differential operator

$$D = P\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right).$$

Prove that no nonzero polynomial solution u of the equation $Du=0$ can be divisible by $P(x_1, \cdots, x_n)$.

4966. *Proposed by G. Di Antonio, Duquesne University, Pittsburgh, Pa.*

Find the equation of all surfaces all of whose normals intersect a given line, say $x=y=z$.

4967. *Proposed by D. S. Mitrinovitch, University of Belgrade, Yugoslavia*

Evaluate the following definite integral

$$J(a, n, k) = \int_{-\pi}^{+\pi} \frac{\cos(n-k)x}{(1 - 2a\cos x + a^2)^n}\, dx.$$

Here n and k are positive integers, and a is a real constant for which the integral has meaning.

4968. *Proposed by Basil Gordon, University of California, Los Angeles*

If M is any square matrix, let $\phi_M(\lambda)$ denote its characteristic polynomial, and M_i the matrix obtained by deleting the ith row and ith column of M. Show that if A and B are $n \times n$ matrices over a field of characteristic 0 or $p \geqq n$ with

$\phi_{A_i}(\lambda)=\phi_{B_i}(\lambda)$ $(i=1, \cdots, n)$, then $\phi_A(\lambda)$ and $\phi_B(\lambda)$ differ by a constant.

4969. *Proposed by David Gale, Brown University, and D. J. Newman, Yeshiva University*

Let C be a closed convex curve, p a measurable function defined on C such that $0 \leqq p \leqq 1$. Prove there is a chord of C whose length ζ satisfies $\int_C p dz = \zeta$.

4970. *Proposed by S. W. Golomb, Jet Propulsion Laboratory, California Institute of Technology*

Let $Q=\{q_i\}$ be an infinite subset of the primes, and let $g(x)$ denote the number of members of Q which do not exceed x. Call Q rare if $\sum 1/q_i$ converges, and call Q sparse if $g(x)=o(x/\log x)$ as $x \to \infty$. What causal relation, if any, exists between rarity and sparsity?

SOLUTIONS

Analytic Solutions of a Functional Equation

4910 [1960, 499]. *Proposed by J. S. Frame, Michigan State University*

Find all analytic solutions of the functional equation $f(2z)=2f(z)f'(z)$.

I. *Solution by J. W. Ellis, Louisiana State University, New Orelans.* The given equation can be differentiated repeatedly to give, for all $n \geqq 0$,

$$2^{n-1}f^{(n)}(2z) = \sum_{q=0}^{n} \binom{n}{q} f^{(q)}(z) f^{(n-q+1)}(z).$$

Clearly the origin is not a pole. Straightforward induction arguments will then give the following:

(a) If $f(0)=k \neq 0$, then $f^{(n)}(0)=k(2k)^{-n}$ for all n. Thus $f(z)=k \exp(z/2k)$.

(b) If $f(0)=0$, then $f^{(2n)}(0)=0$ for all n.

(i) If $f'(0)=0$, then $f^{(2n+1)}(0)=0$, so that $f(z) \equiv 0$.

(ii) If $f'(0) \neq 0$, it must equal 1, and $f^{(2n+1)}(0)=Q^n$, where Q is an arbitrary complex number. If $Q=0$, this means $f(z)=z$; if $Q \neq 0$, then $f(z)=c^{-1}\sin cz$, where $c^2=-Q$.

Direct substitution shows that these four types of function satisfy the given equation.

II. *Solution by David Zeitlin, Remington Rand Univac, St. Paul, Minnesota.* This problem is not new. See E. Beke, *Matematikai és Fizikai Lapok*, 48 (1941), pp. 387–392. This reference (with solutions is given in I) is cited in *Differentialgleichungen Lösungsmethoden und Lösungen*, E. Kamke, Third Edition, Chelsea, v. I, p. 660, eq. 10.6b).

Also solved by Stephen Andrea, Alan Beal, Robert Breusch, W. G. Brown, L. Carlitz, P. E. Chernoff, George Glauberman and Burton Fein, Michael Goldberg, Reuben Hersh, W. S. Lawton, Y. Matsuoka, J. G. Mauldon, D. L. Muench, D. J. Newman, Arnold Singer, Vencil Skarda, J. H. van Lint, J. S. White, and the proposer.

An Extension of Picard's Theorem

4911 [1960, 596]. *Proposed by Peter Ungar, New York University*

Let $f(z)=\lambda z+a_2z^2+\cdots$ be an entire function, where $|\lambda|>1$. Let N be any neighborhood of the origin. The images of N under successive iterations of f cover the whole plane with the possible exception of one point. The same is true if $\lambda=1$ and $f(z)\not\equiv z$.

Solution by J. H. van Lint, Technical University, Eindhoven, Netherlands. The iterations f_n of f are all regular on N. If there were two values a and b not taken by any of the functions f_n then these functions would form a normal family (Montel's theorem). Let C be a closed circle $|z|\leqq r$ which lies entirely in N. Since $f_n(0)=0$ for all n, the normal family (f_n) must be uniformly bounded in C (cf. Golusin, *Geometrische Funktionentheorie*), *i.e.* $|f_n(z)|\leqq M$ for $|z|\leqq r$ and all n. Now Cauchy's inequality gives

$$|\lambda^n|\leqq\frac{M}{r}\text{ for all }n\text{ in case }f_n(z)=\lambda^n z+\cdots,$$

$$n|a_\nu|\leqq\frac{M}{r^\nu}\text{ for all }n\text{ in case }f_n(z)=z+na_\nu z^\nu+\cdots\ (a_\nu\neq 0).$$

These inequalities contradict $|\lambda|>1$ and $a_\nu\neq 0$, respectively. This proves that there is at most one value a, not taken by any of the f_n on N.

Also solved by W. A. Veech and the proposer.

Range of Three Integers where a_1 Divides a_2a_3

4912 [1960, 596]. *Proposed by Paul Erdös, Technion, Haifa, Israel*

Prove: there exists an absolute constant $c_1>0$ such that, for every n, there always exist integers a_i $(i=1, 2, 3)$, $n<a_i<n+n^{1/2}+c_1n^{1/4}$, for which a_1 is a divisor of a_2a_3. If $c_2>0$ is sufficiently small this is false; in fact, there are infinitely many values of n such that, for every triplet, $n<a_i<n+n^{1/2}+c_2n^{1/4}$ we must have that a_1 does not divide a_2a_3.

Solution by Koichi Yamamoto, Kyushu University and the University of Southern California. We prove the

THEOREM. (A) *If n is a positive integer, we can find a triplet a_1, a_2, a_3 of integers such that*

$$n<a_1<a_2<a_3<n+n^{1/2}+3n^{1/4},\qquad a_1\mid a_2a_3.$$

(B) *If ϵ is a positive constant, then there exist infinitely many positive integers n such that, for every triplet a_1, a_2, a_3 with*

$$n<a_i<n+n^{1/2}+(3-\epsilon)n^{1/4}\qquad(i=1, 2, 3),$$

we have that $a_1\nmid a_2a_3$.

Proof of (A). For a given n denote by m the integer satisfying

$$2(n+1)^{1/2} < m \leqq 2(n+1)^{1/2} + 1, \tag{1}$$

and by k the integral solution of

$$(m^2-4n)^{1/2} - 2 \leqq k < (m^2-4n)^{1/2}, \qquad k \equiv m \pmod 2. \tag{2}$$

The condition (1) implies

$$m^2 - 4n \geqq 5, \tag{3}$$

which assures the existence of a positive solution k of (2). Put $b=\frac{1}{2}(m+k)$, $c=\frac{1}{2}(m-k)$, $a_1=bc$, $a_2=(b+1)c$, $a_3=b(c+1)$. Then it is evident from $k>0$ that $a_1<a_2<a_3$ and $a_1 \mid a_2a_3$.

It follows from (2) that $4a_1=m^2-k^2>4n$, and so, $a_1>n$.

Before proving $a_3<n+n^{1/2}+3n^{1/4}$, note the following consequences of (1) and (2):

$$m \leqq 2n^{1/2} + n^{-1/2} + 1, \qquad m^2 - 4n \leqq 2m + 3, \tag{4}$$

$$k \leqq (m^2-4n)^{1/2} \leqq (2m+3)^{1/2} \leqq 2n^{1/4} + \tfrac{1}{2}n^{-3/4} + \tfrac{5}{4}n^{-1/4}. \tag{5}$$

Noticing that $\left|(m^2-4n)^{1/2}-3\right| \leqq k-1$, except for the case where $m^2-4n=5$, and using (4) and (5), we have

$$\begin{aligned} 4a_3 &= (m+1)^2 - (k-1)^2 \\ &\leqq (m+1)^2 - ((m^2-4n)^{1/2}-3)^2 = 4n + 2m - 8 + 6(m^2-4n)^{1/2} \\ &\leqq 4n + 4n^{1/2} + 12n^{1/4} - 6 + 2n^{-1/2} + \tfrac{15}{2}n^{-1/4} + 3n^{-3/4}. \end{aligned}$$

We see that $2n^{-1/2}+\frac{15}{2}n^{-1/4}+3n^{-3/4}<6$ for $n \geqq 9$. Thus $a_3<n+n^{1/2}+3n^{1/4}$ for $n \geqq 9$, except for the case where $m^2-4n=5$. In this case we have $k=1$ and

$$\begin{aligned} 4a_3 &= (m+1)^2 = 4n + (m^2-4n) + 2m + 1 = 4n + 2m + 6 \\ &\leqq 4n + 4n^{1/2} + 8 + 2n^{-1/2} < 4n + 4n^{1/2} + 12n^{1/4}. \end{aligned}$$

It is easy to check $a_3<n+n^{1/2}+3n^{1/4}$ directly, for $n \leqq 8$. This completes a proof of (A).

Proof of (B). Let $0<\epsilon<\frac{1}{4}$ and let δ be a rational number such that

$$1 + \tfrac{1}{4}\epsilon < \delta < 1 + \tfrac{3}{4}\epsilon, \tag{6}$$

and consider a family of positive integers q such that δq is integral. We confine ourselves to numbers n of the form

$$n = m^2, \qquad m = q^2 - \delta q \tag{7}$$

and prove that there are infinitely many numbers q such that

$$m^2 < a_i < m^2 + m + (3-\epsilon)m^{1/2} \qquad (i = 1, 2, 3), \tag{8}$$

with distinct a_i, entails $a_1 \nmid a_2a_3$. We prove first the

LEMMA. *Suppose q is sufficiently large. If there exists a triplet of integers a_i such that $a_1 \mid a_2a_3$ and the condition* (8) *is satisfied, then there exist two positive integers t and v such that*

$$1 \leqq t \leqq 8, \tag{9}$$

$$v - \tfrac{1}{2}t + \tfrac{1}{2}t^{1/2} < t^{1/2}q < v - \tfrac{1}{2}t + \tfrac{1}{2}t^{1/2} + \frac{3 - t^{1/2}}{2t^{1/2}} + \frac{(t-1)\epsilon}{2t^{1/2}}. \tag{10}$$

Indeed, if a_i satisfy (8) and $a_1 \mid a_2a_3$, then $a_1 = b_1c_1$, $a_2 = b_1c_2$, $a_3 = b_2c_1$. We note that $a_2 < a_1 < a_3$ is impossible because, then, $a_3 - a_2 = (b_2 - b_1)c_1 + b_1(c_1 - c_2) \geqq c_1 + b_1 \geqq 2(b_1c_1)^{1/2} = 2a_1^{1/2} > 2m$ and $a_3 - a_2 < m + (3-\epsilon)m^{1/2}$ imply $3 > m^{1/2}$ which is false for q sufficiently large. Similarly $a_3 < a_1 < a_2$ is impossible. There remain two possibilities: (I) $b_1 < b_2$, $c_1 < c_2$, and (II) $b_1 > b_2$, $c_1 > c_2$, and in either case we can assume $b_1 \geqq c_1$ without loss of generality. Thus $b_1 \geqq (b_1c_1)^{1/2} = a_1^{1/2} > m$. Define v by $b_1 = m + v$. Define u in case (I) by $a_1 = m^2 + u$, and in case (II) by $a_2 = m^2 + u$. Now, in case (I) we have $a_1 + b_1 = b_1(c_1 + 1) \leqq b_1c_2 = a_2$, while in case (II) we have $a_2 + b_1 = b_1(c_2 + 1) \leqq b_1c_1 = a_1$. In either case, $m^2 + u + b_1 = m^2 + m + u + v \leqq$ some a_i; and hence, from (8) we have

$$u + v < (3 - \epsilon)m^{1/2}. \tag{11}$$

Since a_1 or $a_2 = m^2 + u = m^2 - v^2 + v^2 + u$ is a multiple of $b_1 = m + v$, so also is $v^2 + u$ a multiple of b_1:

$$v^2 + u = t(m + v), \qquad t > 0. \tag{12}$$

Since $tm < t(m+v) = v^2 + u \leqq (u+v)^2 < (3-\epsilon)^2 m$, we have $t < 9$, which is (9).

Next, it follows from (7) and (12) that

$$\begin{aligned}(v - \tfrac{1}{2}t)^2 &= v^2 - tv + \tfrac{1}{4}t^2 = tm - u + \tfrac{1}{4}t^2 = t(q^2 - \delta q) - u + \tfrac{1}{4}t^2 \\ &= t(q - \tfrac{1}{2}\delta)^2 + \tfrac{1}{4}t^2 - \tfrac{1}{4}t\delta^2 - u < t(q - \tfrac{1}{2}\delta)^2 + O(1),\end{aligned}$$

as $q \to \infty$. Hence if q is sufficiently large we have

$$\begin{aligned}v - \tfrac{1}{2}t &< t^{1/2}(q - \tfrac{1}{2}\delta) + O(q^{-1}) < t^{1/2}(q - \tfrac{1}{2}\delta) + \tfrac{1}{8}t^{1/2}\epsilon \\ &< t^{1/2}q - \tfrac{1}{2}t^{1/2}(1 + \tfrac{1}{4}\epsilon) + \tfrac{1}{8}t^{1/2}\epsilon = t^{1/2}q - \tfrac{1}{2}t^{1/2},\end{aligned}$$

by using (6). This proves the first inequality in (10). The second inequality is proved similarly by using (6), (7), (12) and (11). Note that $v - (t+1)/2 \geqq 0$, which is necessary for extraction of square roots from the inequality, is a consequence of (12) and (11):

$$\begin{aligned}v(v - (t+1)) &= v^2 - (t+1)v = tm - (u + v) \\ &\geqq tm - (3 - \epsilon)m^{1/2} > m - 3m^{1/2} > 0,\end{aligned}$$

for sufficiently large q. The lemma is thus proved.

Now the lemma guarantees that if q is sufficiently large and if there exists a triplet a_i such that (8) is satisfied and $a_1 \mid a_2a_3$, then the fractional part $\{t^{1/2}q\}$

$=t^{1/2}q-[t^{1/2}q]$ of the number $t^{1/2}q$ lies in the open interval

$$I_t=\left(-\tfrac{1}{2}t+\tfrac{1}{2}t^{1/2},\ -\tfrac{1}{2}t+\tfrac{1}{2}t^{1/2}+\frac{3-t^{1/2}}{2t^{1/2}}+\frac{(t-1)\epsilon}{2t^{1/2}}\right)\pmod 1$$

for some t with $1\leqq t\leqq 8$. If $t\neq 1$, then the length of I_t is

$$l_t=\frac{3-t^{1/2}}{2t^{1/2}}+\frac{(t-1)\epsilon}{2t^{1/2}}<1.$$

Note in particular $I_1=(0,\ 1)$, $I_4=(0,\ \frac{1}{4}+\frac{3}{4}\epsilon)\pmod 1$, and hence the condition

$$\{t^{1/2}q\}\notin I_t \tag{13}$$

is satisfied for $t=1$ and $t=4$. We note further, since $\epsilon<1/4$, that $l_2+l_8<1$.

Now, since the five irrational numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$ are linearly independent with respect to the rational number field, we find, by a theorem of H. Weyl (*Math. Ann.*, vol. 77, 1916, pp. 313–352, Satz 4, p. 319) on the multidimensional uniform distribution (mod 1) of irrational numbers, that the "probability" that q, a multiple of the denominator of δ, satisfies the condition (13) for all $t=1,\ 2,\ \cdots,\ 8$ is greater than

$$(1-l_2-l_8)(1-l_3)(1-l_5)(1-l_6)(1-l_7)>0.$$

Thus there exist infinitely many q's for which (13) is true for $t=1,\ 2,\ \cdots,\ 8$, *i.e.* for which

$$(q^2-\delta q)^2<a_i<(q^2-\delta q)^2+q^2-\delta q+(3-\epsilon)(q^2-\delta q)^{1/2}$$

for $i=1,\ 2,\ 3$ entails $a_1\mid a_2a_3$. This proves the assertion (B).

Also solved by Robert Breusch and J. H. van Lint.

Editorial Note. The proposer remarks that it may be shown trivially that for any $c_3>0$ there exist infinitely many n such that three distinct integers a_i can be found satisfying $n<a_i<n+n^{1/2}+c_3n^{1/4}$ and such that $a_1\mid a_2a_3$.

Distinct Products of Pairs of Integers

4913 [1960, 596]. *Proposed by Paul Erdös, Technion, Haifa, Israel*

Let w be any positive integer, and let a_i, a_j be integers satisfying

$$w^2\leqq a_i<(w+1)^2,\qquad w^2\leqq a_j<(w+1)^2.$$

Prove that all products a_ia_j are distinct.

Solution by Jack Silver, Undergraduate, Montana State University. We first show that for $w^2\leqq a_i<(w+1)^2$, $i=1,\ 2,\ 3,\ 4$, $a_1a_2\leqq a_3a_4$ implies $a_1+a_2\leqq a_3+a_4$. If $a_i=w^2+b_i$ and $a_1+a_2>a_3+a_4$, then $b_1+b_2-b_3-b_4\geqq 1$. Therefore $(w^2+b_1)\cdot(w^2+b_2)\leqq(w^2+b_3)(w^2+b_4)$ yields

$$(b_1+b_2-b_3-b_4)w^2\leqq b_3b_4-b_1b_2<(b_1+b_2-b_4)b_4-b_1b_2,$$

$$w^2<(b_1-b_4)(b_4-b_2)\leqq\tfrac{1}{4}(b_1-b_2)^2\leqq w^2,$$

where we have used $b_1-b_2 \leqq 2w$ and the fact that $xy \leqq \frac{1}{4}(x+y)^2$. But this is a contradiction.

Thus, if $a_1a_2=a_3a_4$, then $a_1+a_2=a_3+a_4$, whence $a_1^2+a_3a_4=a_1(a_1+a_2)=a_1(a_3+a_4)$, or $(a_1-a_3)(a_1-a_4)=0$, so that a_1 equals a_3 or a_4. The desired result is now immediate.

The result is sharp. We cannot replace $<$ by $\leqq$ as the counterexample $w^2\cdot(w+1)^2=(w^2+w)\cdot(w^2+w)$ shows.

Also solved by A. C. Aitken, W. J. Blundon, Robert Breusch, L. J. Burton, Bro. Joseph Heisler, J. A. Holbrook, W. H. Jobe, Eileen Jones and Edward Barbeau, John B. Kelly, J. Kestelman, W. S. Lawton, Joe Lipman, D. C. B. Marsh, D. J. Newman, E. J. F. Primrose, J. L. Selfridge, Wu Ta-Sun, J. H. van Lint, Benjamin Weiss.

Generalization of a Familiar Combination Formula

4914 [1960, 596]. *Proposed by Leo Moser, University of Alberta*

For any positive integers m, n, prove that K is an integer, where

$$K = \frac{(mn)!1!2!\cdots(m-1)!1!2!\cdots(n-1)!}{1!2!\cdots(m+n-1)!}.$$

Solution by Leonard Carlitz, Duke University. Put

$$m = ap + r \quad (0 \leqq r < p), \qquad n = bp + s \quad (0 \leqq s < p),$$
$$m + n = (a + b + \epsilon)p + t \quad (0 \leqq t < p,\ \epsilon = 0 \text{ or } 1),$$

where p denotes a prime power. Then

$$\sum_{k=1}^{m-1}\left[\frac{k}{p}\right] = \left[\frac{p}{p}\right] + \cdots + \left[\frac{2p-1}{p}\right] + \left[\frac{2p}{p}\right] + \cdots$$
$$= p + 2p + \cdots + (a-1)p + ra$$
$$= \tfrac{1}{2}pa(a-1) + ra = \tfrac{1}{2}a(pa - p + 2r) = \tfrac{1}{2}a(m - p + r).$$

Hence it will suffice to show that

$$\left[\frac{mn}{p}\right] + \tfrac{1}{2}a(m-p+r) + \tfrac{1}{2}b(n-p+s) \geqq \tfrac{1}{2}(a+b+\epsilon)(m+n-p+t),$$

that is

$$2\left[\frac{mn}{p}\right] \geqq an + bm + (a+b)t - ar - bs + \epsilon(m+n-p+t),$$

or

$$(*) \qquad (a+b)(r+s) + 2\left[\frac{rs}{p}\right] \geqq (a+b)t + \epsilon(m+n-p+t).$$

If $\epsilon=0$, so that $r+s=t$, (*) obviously holds.

If $\epsilon=1$, so that $r+s=p+t$, (*) becomes

$$(a+b)p+2\left[\frac{rs}{p}\right]\geqq(a+b)p+2t,$$

that is $[rs/p]\geqq t$. But since $(p-r)(p-s)>0$, $rs>p(r+s-p)=pt$, so that $[rs/p]\geqq t$, completing the proof.

Also solved by Robert Breusch, L. M. Kaplan, Richard Karch, and K. A. Post.

Note by the proposer. The given expression is a solution to the following problem: In how many ways can the integers $1, 2, \cdots, mn$ be placed in an $m\times n$ matrix so that the numbers in every row and column are monotone increasing? With this interpretation, K must obviously be integral.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Frontiers of Numerical Mathematics. Edited by Rudolph E. Langer. The University of Wisconsin Press, Madison, Wis., 1960. xi+132 pp. $3.50 (Photo-offset)

Eight invited speakers were asked to set forth the mathematical problems that stand astride the advance of their specialities. These papers were delivered at a symposium conducted by the Mathematics Research Center, U. S. Army and the National Bureau of Standards held at the University of Wisconsin in October, 1959. *Frontiers of Numerical Mathematics* contains the eight papers, each followed by the comments of some of the eighteen discussants who were there.

The titles and the authors of the papers are: *Stress Analysis in the Plastic Range*, William Prager; *Some Mathematical Problems of Nuclear Reactor Theory*, Garrett Birkhoff; *Numerical Problems of Contemporary Celestial Mechanics*, Sdenek Kopal; *Aeroelasticity*, Lee Arnold; *Operations Research*, Philip M. Morse; *Mathematical Bottlenecks in Theoretical Chemistry*, J. O. Hirschfelder; *Magneto-hydrodynamics*, S. Chandrasekhar; *The Solution of Systems of Partial Differential Equations Arising in Meteorology*, J. Smagorinsky.

The papers are provocative and deserve scanning for new ideas. The place of the high-speed computer is quite uniformly considered as that of a useful tool following rather than replacing theory. However, Professor Morse mentions that simulation on a computer may be an aid to the development of improved mathematical models.

Advances in technology accentuate difficulties of old problems, as for example the orbit computations of a man-launched satellite, which since its launching "has already completed more revolutions around the earth than the number of

years which has elapsed since the dawn of human civilization."

The papers are not uniform in length nor in detail. However, the variety of views expressed in the well chosen chapters on applied mathematics is most stimulating. The photo-offset printing is well done.

HERBERT A. MEYER
University of Florida

Introduction to Probability and Random Variables. By George P. Wadsworth and Joseph G. Bryan. McGraw-Hill, New York, 1960. vii+202 pp. $8.75.

This book starts with a chapter on preliminary mathematics. It is a very good idea and well done. Often students come ill-prepared, and this way the material or any portion can be taken at will. Also the presentation of techniques does not clutter up the theory. I think such a chapter should be even more complete, including matrices. The lack of matrices throughout the book is a definite hindrance and permits only a limited presentation of multivariate theory at a time when it should be stressed more than before. Bayes's theorem is included, which is good in view of the tendency to omit it from elementary texts. Queueing theory and other modern applications are introduced, illustrating that this book is a new presentation rather than just a rewrite of standard texts.

In conjunction with the presentation of theory there are given many and practical examples, which are excellent. This should help overcome the essential debility in many books at this level, in which, although the theory is presented well, the student gets a "Yes, but what of it?" feeling. A sampling of the exercises shows that they are very good ones. They are appropriate in difficulty, cover the material well, and include some practical applications without becoming trivial.

With an additional hundred pages on the mathematics of statistical methodology, this book would be a competitor at the Hoel-Fraser level for a mathematical statistics course. With a hundred pages less, it would be a competitor for Cramér's elementary text. As it is, it is just appropriate for a mathematics department course for undergraduates as an introduction to probability and random variables, just as the title says. All in all, it is a well-written book and should serve extremely well if used for the appropriate purpose.

ROBERT H. RIFFENBURGH
University of Hawaii

Lectures on the Theory of Functions of a Complex Variable, Vol. I, *Holomorphic Functions.* By G. Sansone and J. C. H. Gerretsen. Noordhoff, Groningen, 1960. xii+481 pp.+index. Dfl. 45.– (paper) or Dfl. 48.75 (cloth).

This readable book is the first part of a two-volume English edition of Sansone's *Lezioni sulla Teoria delle Funzioni di una Variabile Complessa* (1947) with a new text containing revisions and rearrangement of the Italian work. It starts from first principles but does not emphasize logical foundations. The book is

primarily designed for beginners interested in a detailed account of special topics of the classical theory. No attempt is made to present a scholarly work and the reader will notice that a bibliography is not provided. Among the topics discussed are: power series, elementary functions, Cauchy integral theorem, residue theory, Weierstrass factor theorem, Mittag-Leffler theorem, elliptic functions, integral functions of finite order, Dirichlet series, Riemann zeta function, Laplace integral and asymptotic series.

HIROSHI YAMAUCHI
University of Hawaii

Rings of Continuous Functions. By Leonard Gillman and Meyer Jerison. The University Series in Higher Mathematics, Van Nostrand, Princeton N. J., 1960. ix+300 pp. $8.75.

A very interesting book on a subject which offers large vistas for further research and which will undoubtedly receive increasing recognition on the roster of graduate courses offered in our universities. Through a succession of preliminary editions developed in formal seminars and in the more critical atmosphere of office and home, the two authors have over the past 5 years performed the enormous task of the preparation, separation, ordering, and finally presentation of a plenitude of cohesive material. The result is a clear book which allows the reader to reach rapidly good depth and a wide angle view.

Mise en scène: A topological space X, usually completely regular, the ring $C(X)$ of all real continuous functions on X, the ring $C^*(X)$ of all bounded real continuous functions on X. Fundamental problems studied concern the relation of the algebraic properties of the rings (*e.g.* ideal structure) and topological properties of X (*e.g.* compactness). The work of Stone and Čech on compactification is central and is studied in great detail. The elaboration of this material is considered in the middle part of the book. The first third is devoted to the preparation of sharp tools concerning zero-sets, filters, ideals (fixed and free), ordered rings, and residue fields. Then come the chapters on the compactification βX of X, the theorem of Gelfand-Kolmogoroff, and material on the real compact spaces of Hewitt. The final third of the book is devoted to largely independent topics. We find a section on discrete spaces and the relation to nonmeasurable cardinals; a chapter on uniform spaces including Shirota's theorem relating real-compact spaces to spaces with complete uniform structure; and Katětov's work connecting the dimension of X to the analytic dimension of $C^*(X)$.

The sixteen chapters are followed by substantial exercises which are valuable both to the beginner and to the expert. You will find here, for example, the universal Gegenbeispiel, Tychonoff's plank, as well as the eye-filling wonders of the space $\beta N - N$. Modulo the reviewer's congenital enthusiasm it would seem, nevertheless, that this is an *opus magna laude dignum.*

EDGAR R. LORCH
Columbia University

BRIEF MENTION

Mathematical Tables, Royal Society:

Volume V, Representations of Primes by Quadratic Forms. Prepared by Hansraj Gupta, M. S. Cheema, A. Mehta, and O. P. Gupta. Cambridge University Press, 1960. 135+xxiii, $8.50.

Volume VI, Tables of the Riemann Zeta Function. By C. B. Haselgrove, in collaboration with J. C. P. Miller. xxii+80 pp., $9.50.

Volume VII, Bessel Functions, Part III, Zeros and Associated Values. By F. W. J. Oliver. xv+lix+79 pp., $9.50.

Analogue and Digital Computers. By M. G. Say, A. C. D. Haley and W. E. Scott. Philosophical Library, 1960. viii+308 pp., $15.00.

Although the publication date and the price might suggest this was an up-to-date book containing last-minute information, a typical sentence discloses the truth, "A typical size of store is about 10,000 digits (about 320 words of 32 digits, or 256 words of 40 digits)." Page 216.

Proceedings of the 1959 Computer Applications Symposium. Armour Research Foundation, Illinois Institute of Technology, 155 pp., $3.00.

Fourteen invited papers presented on October 28–29, 1959. The first day's papers were devoted to business and management applications, while those on the following day stressed engineering and scientific applications. The interesting discussion of the pros and cons of automatic programming which occurs in the panel discussion makes the book worth investigating in and of itself.

Digital Computers and Nuclear Reactor Calculations. By Ward C. Sangren. Wiley, New York, 1960. xi+208 pp., $8.50.

The book begins with an introduction to reactor problems, numerical analysis, digital computers, and their programming, followed by more detailed discussions of the important high-speed digital reactor calculations.

Theoretical Hydrodynamics. By Milne-Thomson. 4th Ed., Macmillan, New York, 1960. 650 pp., $11.00.

Vector dynamics with a good bit of solid mathematics interspersed among the air foil and hydrodynamic applications. Not a modern book in a strict sense of the word, but a solid one.

Contributions to the Theory of Nonlinear Oscillations. Volume V. Princeton University Press, 1960. 284+append., $5.00. Princeton Annals of Mathematics Studies, Vol. 45.

A collection of advanced papers which will undoubtedly be reviewed separately in *Mathematical Reviews* and other journals.

The Mathematics of Radiative Transfer. By I. W. Busbridge. Cambridge University Press, 1960. 133 pp., $5.00. Number 50 in Cambridge Tracts in Mathematics and Mathematical Physics.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., *University of Buffalo*

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Harvey Cohn, University of Arizona, represented the Association at the inauguration of Dr. G. H. Durham as President of Arizona State University on March 11, 1961.

Professor Mary K. Landers, Hunter College, representated the Association at the inauguration of Dr. L. L. Jarvie as President of the New York City Community College on February 16, 1961.

Mr. F. A. Lee, Jr., Marion Institute, represented the Association at the inauguration of Dr. C. A. Anderson as President of Judson College on February 3, 1961.

Professor J. H. Wahab, Louisiana State University at New Orleans, represented the Association at the inauguration of Dr. H. E. Longnecker as President of Tulane University on April 15, 1961.

Catholic University of America: Dr. Inge Christensen and Mr. Gustav Hensel have been appointed Instructors.

Massachusetts Institute of Technology: Professor I. E. Segal, University of Chicago, has been appointed Professor; Dr. J. G. Glimm, Institute for Advanced Study, and Dr. James Munkres, Princeton University, have been appointed Assistant Professors; Drs. R. B. Darst, Louisiana State University, S. Y. Husseini, and Ronald Jacobowitz, Princeton University, J. I. Richards, Harvard University, and F. S. Van Vleck, University of Minnesota, have been appointed Instructors; Associate Professor G. B. Thomas, Jr., has been promoted to Professor; Assistant Professors H. P. McKean, Jr., and D. B. Ray have been promoted to Associate Professors; Dr. D. J. Benney has been promoted to Assistant Professor; Professors R. D. Douglass and D. J. Struik retired with the title of Professor Emeritus; Institute Professor Norbert Wiener retired with the title of Institute Professor Emeritus; Associate Professor S. D. Zeldin retired with the title of Associate Professor Emeritus.

Michigan State University: Professor J. S. Frame has resigned as Head of the Department of Mathematics retaining the position of Professor of Mathematics; Professor C. P. Wells has been appointed Head of the Department of Mathematics; Assistant Professors J. E. Adney, Purdue University, Ti Yen, Illinois Institute of Technology, and R. P. Gilbert, University of Pittsburgh, have been appointed Assistant Professors; Assistant Professors J. G. Hocking, R. H. Oehmke, and M. L. Tomber have been promoted to Associate Professors.

University of British Columbia: Drs. Mario Benedicty, University of Pittsburgh, C. W. Clark and Maurice Sion, University of California, Berkeley, and Z. A. Melzak, Bell Telephone Laboratories, Murray Hill, New Jersey, have been appointed Assistant Professors; Drs. D. F. Rearick and R. C. Thompson, California Institute of Technology, and Lorraine Schwartz, University of California, Berkeley, have been appointed Instructors; Mr. J. F. Scott-Thomas, Massachusetts Institute of Technology, has been appointed Lecturer; Assistant Professor H. A. Thurston has been promoted to Associate Professor; Mr. Elöd Macskásy and Dr. Rimhak Ree have been promoted to Assistant Professors.

University of Kentucky: Dr. A. L. Duquette, University of Colorado, has been appointed Assistant Professor; Professor Stanislaw Balcerzyk, University of Torun, Poland, has been appointed Research Instructor.

University of Nevada: Assistant Professor Ronald Macauley, University of Washington, has been appointed Assistant Professor; Assistant Professor R. N. Thompson has been promoted to Associate Professor.

Dr. C. H. Boll, Lockheed Aircraft Corporation, Sunnyvale, California, has accepted a position with Loral Electronic Corporation, Tustin, California.

Dr. Evelyn B. Collins, Staff Assistant at the Space Computing Center of International Business Machines, Washington, D. C., has joined the technical staff of Space Technology Laboratories, Los Angeles, California.

Dr. E. H. Crisler, Bendix Aviation Corporation, South Bend, Indiana, has accepted a position as a member of the Senior Staff of Hughes Aircraft Company, Culver City, California.

Dr. W. E. deMalignon, University of South Dakota, has been appointed Assistant Professor at San Diego State College.

Professor J. W. Givens, Wayne State University, has been appointed Professor of Engineering Sciences at Northwestern University.

Mr. L. K. Grodman, International Business Machines, Poughkeepsie, New York, has accepted a position with Mitre Corporation, Bedford, Massachusetts.

Dr. Ali Kyrala, Goodyear Aircraft Corporation, Litchfield Park, Arizona, has been appointed Associate Professor at Arizona State University.

Mrs. Carolyn S. Sharif, Austin Public Schools, Austin, Texas, has been appointed Teacher at Sunset High School, Dallas, Texas.

Professor K. W. Wegner, Carleton College, has been appointed Chairman of the Department of Mathematics to replace Professor K. O. May who has returned to full-time teaching.

Professor Emeritus H. C. Feemster, York College, died in February, 1961. He was a Charter Member of the Association.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the Summer of 1961:

Michigan State University, June 19 to September 1: Professor Larcher, projective geometry, theory of matrices and groups; Professor Hocking, introduction to sets and abstract spaces; Professor Gilbert, theory of functions of a complex variable; Professor Wasserman, partial differential equations. June 19 to July 26: Professor Hertzog, advanced calculus, theory of numbers; Professor Stewart, advanced topics in matrices. July 27 to September 1: Professor Doyle, advanced calculus, topology; staff, advanced differential equations, theory of polynomials.

University of Colorado, June 16 to July 21; July 24 to August 25: Professor Dunton, theory of numbers; Professor Zirakzadeh, foundations of geometry; Professor Schmidt, topology, advanced calculus; Professor Hodges, infinite processes, history of mathematics; Dr. Oliver, teaching of secondary school mathematics; Professor Fischer, first session, Professor Jones, second session, modern algebra; Professor McKelvey, foundations of analysis; Dr. Hirsch, theory of groups.

University of Illinois, June 19 to August 12: Professor Ketchum, functions of real variables, introduction to numerical analysis; Professor Mendel, introduction to higher analysis (real variables); Professor Peters, elementary geometry from a modern viewpoint; Professor Ribenboim, group theory; Professor Blyth, advanced statistics; Professor Rotman, topological spaces. In addition the following courses will be given: fundamental concepts of algebra, linear transformations and matrices, introduction to higher algebra, advanced calculus, differential equations and orthogonal functions, complex variables and application.

University of North Carolina at Chapel Hill, June 6 to July 18: Professor Garner, history of mathematics; Professor Hill, elementary mathematical statistics; Professor MacNerney, linear algebra; Professor Linker, differential equations; Professor Mackie, theory of numbers; Professor Artzy, foundations of geometry; Professor Pettis, some recent results in algebra. July 19 to August 26: Professor Lasley, analytic geometry from higher point of view; Dr. Patty, theory of equations; Dr. Buckholtz, advanced calculus; Professor Wells, topics in analysis; Professor Whyburn, advanced differential equations.

University of Virginia, June 19 to August 12: Dr. Henderson, foundations of geometry, Professor Williams, foundations of algebra, introductory analysis; Professor Malbon, differential equations and applied mathematics; mathematics for teachers; Professor Paige, advanced analysis.

University of Wisconsin, Professor Sanderson, applied differential equations, elementary plane topology; Professor Losey, applied mathematical analysis, projective geometry; Professor MacDuffee, matrices and their applications, advanced topics in algebra; Professor Immel, higher analysis, introduction to the theory of probability; Professor Rothman, advanced calculus, modern views of mathematics; Professor Struble, introduction to measure and integration; Professor Jones, advanced topics in point-set topology; Professor Davis, theory and operation of computing machines, advanced topics in real variable theory.

West Virginia University, June 12 to July 21: Professor Cochran, astronomy for teachers, fourier series and partial differential equations; Professor Cunningham, advanced calculus; Professor Peters, introduction to algebraic theories, group theory. July 24 to August 30: Professor Bragg, theory of numbers, linear algebra; Mrs. Easton, special topics; Professor Stewart, advanced calculus, higher plane curves; Professor Vest, operational methods in partial differential equations.

TRAVEL GRANTS FOR ATTENDANCE AT THE INTERNATIONAL CONGRESS OF MATHEMATICS

Travel grants will be made to a number of mathematicians who wish to attend the International Congress of Mathematicians in Stockholm, on August 15–22, 1962. It is hoped that funds available through various sources may provide travel assistance for a considerable number of mathematicians.

There will be a greater effort than in the past to give aid to younger people. As grants will be made only to those who have filed applications, it is urgent that any who wish to receive a grant should fill out and file an application. Younger people are urged to file applications so that their cases can be considered. Applications can be obtained from the Division of Mathematics, National Academy of Sciences, National Research Council, Washington 25, D. C. by requesting an application for a travel grant to the 1962 International Congress.

The deadline for filing of applications is November 1, 1961, and an attempt will be made to announce the grants by January 1, 1962. Awarding of grants will be made only to those persons whose applications have been received, in good order, by November 1. The selection will be made by a committee consisting of the regular Committee on Travel Grants of the Division of Mathematics of the National Academy of Sciences—National Research Council enlarged to include representatives of societies affiliated with the Division and representatives of various governmental agencies.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE EDITOR OF THE MATHEMATICS MAGAZINE

The Board of Governors of the Association has elected Professor Robert E. Horton of Los Angeles City College as Editor of the *Mathematics Magazine* for a three-year term beginning January 1, 1961.

Professor Horton has been Editor of the Problems and Questions Department of the *Mathematics Magazine* since 1953 and its Editor-in-Chief since 1959. He has been an engineer with Douglas Aircraft Co. 1938–39, a mathematics teacher at Black-Foxe Military Institute 1939–41, a mathematics teacher in the Los Angeles City Schools in 1941, a mathematics instructor at the University of California, Los Angeles 1946–47, a mathematics instructor at Los Angeles City College since 1947 and since 1960 Assistant Dean of Instruction at Los Angeles City College. He has published numerous articles in the *Mathematics Magazine* and other publications.

HENRY L. ALDER, *Secretary*

THE NOVEMBER MEETING OF THE MINNESOTA SECTION

The annual fall meeting of the Minnesota Section of the Mathematical Association of America was held on November 5, 1960, at the University of North Dakota, Grand Forks, North Dakota. Professor Charles Hatfield, University of North Dakota, presided at the morning session, and the Section Chairman, Professor Fulton Koehler, University of Minnesota, presided at the afternoon session. There were 63 persons registered for the meeting, of whom 44 were members of the Association.

During the brief business meeting, the Section Chairman discussed the arrangements for conducting the 1961 High School Mathematics Contest.

One feature of the day's program was a panel discussion on the use of television in the teaching of mathematics. The panelists were Professors Bernard Derwort, St. Thomas College, John Dyer-Bennet, Carleton College, and Jack Indritz, University of Minnesota. Professor Fulton Koehler was the moderator. The panelists, each of whom had had actual experience with this new medium, agreed that an increased utilization of audio-visual devices is inevitable, but split on whether this is desirable, with Professor Indritz tending to welcome the new techniques, Professor Dyer-Bennet being quite critical, and Professor Derwort remaining somewhat neutral.

The following papers were presented:

1. *Mathematics in Taiwan*, by Professor K. W. Wegner, Carleton College.

Describing his year as a Fulbright Lecturer at National Taiwan University in 1959–60, the speaker mentioned the lack of any graduate offerings in mathematics in Taiwan, the small number of teachers with advanced degrees, the high standards in the courses given, the intense competition for admission to college, the great difficulties encountered by students in getting to foreign countries for graduate work, the low salaries of teachers, and the activities of the research organization Academica Sinica. He pointed out that the basic sciences and mathematics are losing the best students to engineering because of the lack of opportunities at graduation.

2. *On approximation of alternating series*, by Professor O. E. Stanaitis, St. Olaf College.

The remainder of an alternating series is evaluated by special inequalities obtained from Euler's summation formula. This leads to fast, accurate approximations of alternating series. It has been shown, for example, that the series $S=\sum_{n=1}^{\infty}(-1)^{n+1}/n^3$ can be approximated correctly to six decimal places by adding only ten terms of the series. In order to secure the same accuracy without evaluation of the remainder it would be necessary to calculate a thousand terms of the series.

3. *A Monte Carlo experiment*, by Mr. R. M. Collins, Jr., Minnesota Mutual Life Insurance Co., St. Paul, Minnesota.

The Monte Carlo technique was used to simulate claim experience in a group of lives. To simulate the "exposure" of each life, a random number was compared with the probability of death. If the number was less than or equal to that probability, a death occurred. This was done for the entire group n times to simulate n years of exposure. The result was a claim distribution for the group. The random number supply consisted of an initial table of 1000 numbers from which succeeding tables were generated by an accumulation process. The experiment was performed using a Datatron 205.

4. *Inner and outer quotients*, by Professor Joong Fang, St. John's University, Collegeville, Minnesota.

Some authors in elementary analysis venture to present complex quotients as vectors, but all authors in vector analysis fail to mention "vector quotients" which, however, do exist, and are found to be as follows: 1. The inner quotient: $B=cA/|A|^2+C\times A$ where A, B, and C are vectors, and $A\cdot B=c$ (c a scalar); in particular, if $c=1$, then $B=A^{-1}$; *i.e.*, $A\cdot A^{-1}=1$. 2. The outer quotient: $B=C\times A/|A|^2+cA$ if and only if $A\cdot C=0$, where $A\times B=C$, $(A\neq 0)$ and c is a scalar.

5. *Some singular cases of the implicit function theorem*, by Professor W. S. Loud, University of Minnesota (by invitation).

Singular cases of the implicit function theorem, in which the fundamental Jacobian is zero, arise in the study of periodic solutions of perturbed differential equations. The problem of solving the system $F(x, y, z)=G(x, y, z)=0$ for x and y as functions of z is treated, where at $(0, 0, 0)$, $F=G=F_xG_y-G_xF_y=0$. The cases are analyzed in which a knowledge of the partial derivatives of F and G at $(0, 0, 0)$ through order three is sufficient to resolve the problem of existence and local behavior of solutions. The technique can be extended to more general systems.

6. *A definition of functions on the strength of an operative logic*, by Professor Joong Fang, St. John's University, Collegeville, Minnesota.

This is a sequel to *Inverse functions vs. "converse" functions* (this MONTHLY, vol. 66, p. 947), which is further founded here on an operative logic (in the spirit of E. Lorenzen's *Einführung in die operative Logik und Mathematik*). The *order* of operators with respect to operands is articulated by the proto-logical "sense" of the relation with respect to its "referent" and "relatum," (in the spirit of B. Russell's *Introduction to Mathematical Philosophy*). A function as such, (*i.e.*, an *operative* set of any logical type) is then generally denoted by a set of dyadic n-tuples, $(x_1, \cdots, x_n)$, where the elements of the set, each of which represents a set of the first type, are always grouped into two in compliance with the principle of *ordered* one-to-one correspondence.

MURRAY BRADEN, *Secretary*

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The twenty-third annual meeting of the Northern California Section of the Mathematical Association of America was held at San Jose State College, January 14, 1961. Professor S. P. Hughart, Chairman of the Section, presided at the afternoon session and at Session I (research papers) in the morning. Professor D. W. Blakeslee, Vice-Chairman of the Section, presided at Session II (papers on the teaching of mathematics). There were 150 persons in attendance, including 110 members of the Association.

At the business meeting the resignation of Professor Roy Dubisch as Secretary-Treasurer was accepted and a resolution was passed expressing appreciation for his services during the past years. Professor B. J. Lockhart, U. S. Naval Postgraduate School, was elected Secretary-Treasurer for a three-year term and Professor G. E. Latta, Stanford University, was elected Vice-Chairman. The Chairman for next year is Professor D. W. Blakeslee, San Francisco State College, and Professor S. P. Hughart, Sacramento State College, is the Program Chairman.

By invitation of the Section, two addresses were given. Professor Daniel Zelinsky, Northwestern University and University of California, Berkeley, spoke on *Functors*, and Professor Günter Lumer, Stanford University, spoke on *Differential Equations: Heuristics and Modern Methods*.

The following papers were presented:

1. *On the construction of T-forms*, by Professor Dmitri Thoro, San Jose State College.

Theorems are obtained for the construction of T-forms with given invariants and leading coefficient. These results are then used to determine integers represented by certain indefinite ternary quadratic forms.

2. *On a semantic construction of intuitionistic logic*, by Professor V. H. Dyson, San Jose State College.

On the basis of a Gentzen-type formalization of Heyting's first order predicate calculus, PC_H^1, a notion of validity is introduced with respect to binary trees. This is shown to be equivalent to the topological interpretation over the totality of open subsets of Cantor's discontinuum. Then an intuitionistically correct proof is given for the equivalence between the weak completeness of PC_H^1 and a certain second order formula. This formula seems to be of basic significance in various contexts with intuitionistic logic and analysis. The extent of its validity is as yet unknown. Thus the major open problem resulting from this paper is an investigation of intuitionistically meaningful notions of a function in the light of this formula.

3. *The Diophantine equation* $x^2+y^2+z^2=m^2$, by Mr. R. S. Spira, University of California, Berkeley.

This equation was first solved by L. E. Dickson, giving the parametric solution: $x=2(ut-vw)$, $y=2(uw-vt)$, $z=u^2+v^2-w^2-t^2$, $m=u^2+v^2+w^2+t^2$. Skolem (1941) gave an incorrect algorithm to find u, v, w, t given x, y, z, m. A correct form of this algorithm is given in this paper. Steiger (1956) empirically found a set of seven conditions on the parameters to assure each primitive solution being obtained once and only once. It is shown in this paper that these conditions can be satisfied using the correct form of Skolem's algorithm, but that the uniqueness proof involves a difficult unsolved problem connected with Brahmagupta's product formula: $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$.

4. *An experimental test of an heuristic suggestion due to Pólya*, by Professor C. M. Larsen, San Jose State College.

In *How to Solve It*, G. Pólya presents a list of heuristic suggestions for solving problems, including the following: Do you know a related problem? Can you use its result? Can you use its method? To investigate the utility of this hint, two sets of problems and solutions were prepared. One set made explicit reference to the hint; the second set, otherwise the same, did not. These materials were then given to matched pairs of students. Their scores on the problems were approximately equal, but students who had the hint explicitly available solve the problems in significantly less time.

5. *Some remarks on definition by induction*, by Professor C. A. Hayes, University of California, Davis.

The principle of definition by induction is one usually glossed over rather casually, yet it is of great importance and should be thoroughly understood by high school mathematics teachers. The speaker's aim is to cast some light into this corner of mathematics and perhaps provide teachers with an approach suitable for presentation to bright high school students.

6. *Inequalities in calculus*, by Professor L. H. Lange, San Jose State College.

Several examples of the use of some traditional inequalities in solving extremum problems without the formal machinery of calculus are discussed.

7. *Change in student attitudes towards mathematics in the last five years*, by Professor D. A. Norton, University of California, Davis.

A questionnaire given five years ago to about five hundred entering freshman at the University of California at Davis testing several hypotheses concerning the formation of attitudes towards mathematics was repeated this fall on a group of about eight hundred entering freshman for the purpose of testing change in attitudes. The study revealed, as expected, that five years of propaganda has produced a decided positive shift in the student attitudes toward mathematics. This shift was considerably greater among the male students than among the female.

8. *The formula for inverting a matrix, or, when is an adjoint not an adjoint*, by Dr. A. B. Novikoff, Stanford Research Institute.

The formula for the entries of A^{-1}, where A is a nonsingular $n \times n$ real matrix is one of a class of similar formulas all involving (i) pattern of plus and minus ones; (ii) the computation of minors; (iii) the taking of a transpose; and (iv) one lone operation of division, namely by the determinant of A to some power. In the special case of the formula for A^{-1}, an auxiliary matrix of "cofactors" is introduced and called (by some authors) the "adjoint" of A. These formulas are all derived by exterior algebra, and it is shown how the transpose operation results.

ROY DUBISCH, *Secretary*

PROPOSED AMENDMENTS TO THE BY-LAWS OF THE M.A.A.

At the business meeting of the Association to be held at Oklahoma State University in Stillwater, Oklahoma, on Wednesday, August 30, 1961, motions will be made to amend the By-Laws as follows:

A. That ARTICLE II be amended to read:

1. There shall be two classes of members, ordinary and institutional.

2. Any person interested in the field of collegiate mathematics shall be eligible for election to ordinary membership in the Association.

3. Any institution, academic or corporate, interested in the support of collegiate mathematics shall be eligible for election to institutional membership in the Association.

4. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission. In the case of individuals the application shall be endorsed by two ordinary members of the Association.

5. (The same as ARTICLE II, Section 3, in the present By-Laws).

B. That the last two words of ARTICLE VI, Section 5, be revised to read: "ordinary members."

C. That ARTICLE VII be amended to read:

1. Ordinary members of the Association shall pay an initiation fee of two dollars ($2) at the time of election. The Board of Governors may authorize the admission to ordinary membership of individuals and classes of applicants without payment of the initiation fee.

2. The annual dues of each ordinary members shall be five dollars ($5), including a subscription to the official journal.

3. The fees, dues, and privileges of institutional members of the Association shall be established from time to time by the Board of Governors.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, that member shall be dropped from the list after due notice.

5. (The same as ARTICLE VII, Section 4, in the present By-Laws.)

6. Any ordinary member who . . . (otherwise the same as ARTICLE VII, Section 5, in the present By-Laws).

HENRY L. ALDER, *Secretary*

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

(As amended to February 1, 1961)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs, and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Election to membership shall be by vote of the Board upon written application from the individual seeking admission endorsed by two members of the Association.

3. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary, a Treasurer, and an Associate Secretary.

2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected members (hereinafter called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision. All members of the Board shall hold over until their respective successors are selected or appointed and qualify.

3. There shall be an Executive Committee, advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor, the Secretary and the Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Assocation, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or of interest to the Association. This committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Governors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This committee shall consist of four members, including the Secretary and the Treasurer.

8. (a) The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by the membership in the Sections of the Association or by the membership in constituencies authorized by the Board for territory where Sections do not exist.

(b) The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

(c) The membership in each Section shall elect triennially a Governor for a term of three years. For these elections, at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section. A Governor who has moved permanently from the Section by which he was elected shall be considered to have ended his term of office on the Board. If the Governor has moved from the Section because he is no longer employed there, it shall be interpreted that he has moved permanently from the Section.

(d) The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary, a Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary and the Treasurer) for four years.

(e) The President shall be ineligible for reëlection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

(f) Elections by the Board shall be made from nomination by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-President and the members of the Finance Committee, and the Board may in any case reject all nominations made and call for a new list.

(g) The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined by a Nominating Committee to be appointed annually for that purpose by the President with the approval of the Board. Approximately six months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Nominating Committee shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Nominating Committee shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

9. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

10. In the absence of the President, the First Vice-President (or in his absence the Second Vice-President) shall have and exercise the powers of the President. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

11. The Secretary shall have the usual duties pertaining to his office, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and special meetings and the giving of due notice of all regular and special meetings of the Association and of the Board of Governors. The Secretary shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary and verified by oath of the President.

12. The Treasurer shall have the usual duties pertaining to his office including the collection of dues and the supervision and sefekeeping of the funds of the Association.

13. (a) There shall be an Executive Director who shall be a paid employee of the Association. He shall have charge of the central office of the Association and shall carry out such other duties as may be assigned to him by the Board. He shall be responsible to the Board and shall attend meetings of the Board, the Executive Committee, and the Finance Committee, but he shall not be *ex officio* a member of these bodies.

(b) The Executive Director shall be elected by the Board under terms and conditions of employment fixed by the Finance Committee.

Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The Board shall hold a meeting each year immediately preceding the annual meeting of the Association. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted. The by-laws of each Section when organized and any subsequent changes in these by-laws must be approved by the Board. The Board shall maintain general supervision over the activities of all Sections.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections except as the Board may provide.

Article VI—Official Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to nonmembers for less than the annual dues of individual members.

Article VII—Dues

1. Members of the Association shall pay an initiation fee of two dollars ($2) at the time of election. The Board of Governors may authorize the admission to membership of individuals and classes of applicants without payment of the admission fee.

2. The annual dues of each member shall be five dollars ($5), including a subscription to the official journal.

3. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

4. New members entering the Association after April 1 of any year shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

5. Any member who because of age is no longer in active service, who is in good standing at the time of his retirement and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from the payment of dues, with the privilege of obtaining the official journal at an annual cost of two dollars ($2).

Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session, thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting. The Secretary shall give such due notice when so instructed by a vote of the Board of Governors or when so petitioned by at least forty members of the Association.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

THE EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Summer Meeting at Oklahoma State University, Stillwater, Oklahoma, on August 29, 30 and 31, 1961. The Register will be conducted in the Terrace Room from 9:00 to 5:00 on each of these three days.

There is no charge for registering either to job applicants or to employers, except when the Late Registration Fee for employers is applicable. Provision will be made for anonymity of applicants upon request and upon payment of $1 to defray the cost involved in handling anonymous listings.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, for application forms and for position description forms, which must be completed and returned to Providence not later than August 4, 1961, in order to be included free of charge in the listings at the meeting in Stillwater, Oklahoma. Forms which arrive after this closing date, but before August 14, will be included in the Register Display at the meeting for a Late Registration Fee of $3.00, and will also be included in the printed listings, but not until ten days after the meeting. The printed listings will be available for distribution both during and after the meeting.

It is essential that applicants and employers register at the Employment Register Desk promptly upon arrival at the meeting to facilitate the arrangement of appointments.

CALENDAR OF FUTURE MEETINGS

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–30, 1961.

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, Louisiana, February 16–17, 1962.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY
NORTHEASTERN, University of Vermont, Burlington, June 20, 1961.
NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.
PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Long Beach State College, March 9, 1962.
SOUTHWESTERN
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